Measuring Distances Between Paving Simplicial Complexes

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Departamento de Matemática
2018

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Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Presidente do Júri,

Porto, ______/______/_________

MSc

Acknowledgments

To my sister for being the most amazing person I have ever met! For making me laugh and for always laughing with me. For all the crazyness together. For being older than me sometimes. Thank you!

To my parents for everything! For believing in me and in apparently crazy second chances. For all the support. For helping me dealing with hard times. For constantly showing me how to be tolerant and a better person. For making me have a better understanding of the world. For helping me seeing the best possible version of the world. Thank you so much!

To the dearest friends ever! To Raquel for all the coffee talks and mind-cleaning walks. For cheering me up every time I needed to. For patiently sharing with me her straightforward and clear way of thinking. For being always a big big friend. To Ana, Ricardo, Patrícias, Joana, Lili, André, João, Guilherme... for just being amazing and open-minded people. For simply being there over and over again.

To all the Professors for giving me new skills, new perspectives and new ways of thinking.

To Professor Pedro for constantly teaching me how to think deeper and clearer. For always finding the most amusing and amazing way of talking about every single concept or idea. For the hard attempt to make me see the world optimistically. For all the incredible help during this work.

Resumo

A noção de matróide constitui uma generalização amplamente estudada do conceito de independência linear de vectores. A presente monografia lida com uma classe estritamente mais ampla de complexos simpliciais cujos elementos admitem uma representação através de uma matriz com entradas no semi-anel booleano, daí serem designados booleanamente representáveis. Começamos por introduzir os conceitos de representação de um complexo simplicial através de uma matriz (booleana) e através de um reticulado; mostramos que, mediante certas condições estes conceitos são equivalentes; e apresentamos formas canónicas de ambos os tipos de representação.

De seguida introduzimos duas possibilidades para quantificar distâncias entre complexos simpliciais booleanamente representáveis que se baseiam na existência de um tipo particular de matrizes booleanas (ditas 1-completas). Estas noções de distância motivaram as questões principais discutidas neste trabalho: obter uma caracterização dos complexos simpliciais que admitem representações booleanas 1-completas alternativas com conjuntos de linhas disjuntos; obter majorantes para a distância entre matróides uniformes.

Palavras-Chave: Complexo Simplicial, Complexo Simplicial Booleanamente Representável, Matróide

Abstract

The notion of matroid constitutes a widely studied abstract generalization of the concept of linear independence of vectors. This monograph deals with a strictly wider class of simplicial complexes whose elements admit a representation through a matrix with entries in the boolean semiring, so they are said boolean representable. We start by introducing the concepts of (boolean) matrix representation and lattice representation of a simplicial complex; we show that, under certain assumptions, they are equivalent; and we present canonical forms of both representations.

We introduce then two possibilities for measuring distances between boolean representable simplicial complexes relying on the existence of a particular type of boolean matrices (said 1-complete). These notions of distance motivated the main questions discussed in the present work: getting a characterization of simplicial complexes admitting alternative 1-complete boolean representations with disjoint sets of lines; and getting upper bounds to the distance between uniform matroids.

Keywords: Simplicial Complex, Boolean Representable Simplicial Complex, Matroid

Contents

Notation Index

Chapter 1

Introduction

The present monograph deals with an emerging class of simplicial complexes (said boolean representable) first introduced in [1], [2] and [3] and largely discussed in [6]. A simplicial complex is a pair $\mathcal{H} = (V, H)$ where V is a finite nonempty set and $H \subseteq 2^V$ is a family containing all the singletons and satisfying the following property: if $X \in H$ and $Y \subseteq X$ then $Y \in H$ (we say that H is closed under taking subsets). The dimension of a simplicial complex $\mathcal{H} = (V, H)$ is defined as $\dim \mathcal{H} := \max\{|X| \mid X \in H\} - 1$. Some results of this monograph focus on a narrower class of simplicial complexes. We say that a simplicial complex $\mathcal{H} = (V, H)$ is paving if H contains every subset $X \in 2^V$ such that $|X| \leq \dim \mathcal{H}$. We are interested in simplicial complexes up to isomorphism.

A simplicial complex $\mathcal{H} = (V, H)$ is a matroid if it satisfies an aditional axiom, known as exchange property:

For all $X, Y \in H$ with $|Y| = |X| + 1$, there exists $y \in Y \setminus X$ such that $X \cup \{y\} \in H$

Note that, given a matrix over a field, the family of all linearly independent sets of column vectors is closed under taking subsets and satisfies the exchange property. So the matroid concept, introduced in [7], constitutes an abstract generalization of the notion of linear independence of vectors and is widely studied. However, not all matroids arise from a matrix over a field [5]. Instead, we may consider matrices over the boolean semiring $\mathbb{B}$ and a new concept of independence of columns of such matrices allowing the family of all independent sets of columns to be closed under taking subsets. We say that a simplicial complex is boolean representable if it arises from a matrix over $\mathbb{B}$. This new notion of independence constitutes a successful attempt to get a matricial representation for all matroids, because every matroid is boolean representable. In fact, this is a strictly wider class of simplicial complexes.

In order to reach this new notion of independence we shall consider matrices up to congruence, meaning that the order of rows and columns in the matrix is not taken into account. Now, given a matrix M over $\mathbb{B}$ with columns indexed by a set V, we say that a subset $X \subseteq V$ is independent with respect to M if M admits a square submatrix with columns indexed by X that, up to congruence, is lower unitriangular, i.e. of the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ ? & 1 & 0 & \dots & 0 \\ ? & ? & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & ? & ? & \dots & 1 \end{pmatrix}$$

The family of independent subsets of V with respect to M (said H_M) is in fact closed under taking subsets and therefore the pair $\mathcal{H}_M = (V, H_M)$ is a simplicial complex; the matrix M is said a boolean representation of every simplicial complex isomorphic to $\mathcal{H}_M$. The simplicial complexes (up to isomorphism) arising this way from a matrix over $\mathbb{B}$ consitute the class of boolean representable ones.

Another notion of representation of a simplicial complex involves lattices (we only deal with finite ones throughout this work). Given a finite lattice L and a subset $A \subseteq L$, we say that L is $\vee$-generated by A if $L = \{\vee X \mid X \subseteq A\}$ and express this fact by the notation (L, A). Then we may consider the family $H_{(L,A)}$ of all subsets $X \subseteq A$ admitting an enumeration $X = x_1 \dots x_k$ such that $(x_1 \vee \dots \vee x_i) < (x_1 \vee \dots \vee x_{i+1})$ for every index $i \in \{1, \dots, k-1\}$. The family $H_{(L,A)}$ is closed under taking subsets and so $\mathcal{H}_{(L,A)} := (A, H_{(L,A)})$ is a simplicial complex; the $\vee$-generated lattice (L, A) is said a lattice representation of every simplicial complex isomorphic to $\mathcal{H}_{(L,A)}$.

An important feature of the class of boolean representable simplicial complexes is the equivalence between these two concepts of representation. More precisely, it is possible to construct an explicit bijective correspondence between the set of boolean matrices (up to congruence) and the set of $\vee$-generated lattices (up to lattice isomorphism). Moreover, the simplicial complex represented by a matrix or a lattice is preserved under these inverse maps.

Another interesting aspect about this class is the fact that the lattice of flats (a known lattice representation of every matroid) is still a lattice representation of every boolean representable simplicial complex. Given a simplicial complex $\mathcal{H} = (V, H)$, a flat of $\mathcal{H}$ is a subset $X \in 2^V$ satisfying the following property:

$$\text{Given } Y \in 2^X \cap H \text{ and } p \in V \setminus X, \text{ then } Y \cup \{p\} \in H$$

For example, in the case of field representable matroids, the set of column vectors belonging to a given vector subspace is always a flat. This additionally allows the definition of a boolean matrix $\operatorname{Mat}\mathcal{H}$ with rows indexed by the set of all flats of $\mathcal{H}$ and columns indexed by V. The entry of $\operatorname{Mat}\mathcal{H}$ corresponding to the flat $X \in 2^V$ and the point $v \in V$ is equal to 0 if and only if $v \in X$. The matrix constructed in this way constitutes a boolean representation for every boolean representable simplicial complex. Moreover, every boolean representation of $\mathcal{H}$ is in fact a submatrix of $\operatorname{Mat}\mathcal{H}$.

We focus then on the particular class of paving simplicial complexes, presenting a characterization of matroids that lie in this class: a paving simplicial complex $\mathcal{H} = (V, H)$ of dimension $n \in \mathbb{N}$ is a matroid if and only if, for every subset $X \subseteq V$ with $|X| = n+1$, one of the following conditions holds: X is a simplex of $\mathcal{H}$; or $X \subseteq F$ for some flat F of $\mathcal{H}$. We use this condition to handle two examples of paving simplicial complexes of dimension 2 that are, in fact, matroids.

We shall present some useful notation:

- Given $n, m \in \mathbb{N}$, the uniform matroid $\mathcal{U}_{m,n}$ is, up to isomorphism, the simplicial complex whose points are $1, \ldots, n$ and the simplexes are all subsets of $\{1, \ldots, n\}$ with at most m elements.

- Given a boolean matrix X with rows and columns indexed by sets R and V, respectively, we may write M as an array in the usual way $M = (m_{rv})_{r\in R, v\in V}$. Now, for every element $r \in R$, define the set $Z_r^{(M)} := \{v \in V \mid m_{rv} = 0\}$; and say that $Z_r^{(M)}$ is a line of M if $1 < \left|Z_r^{(M)}\right| < |V|$. Use $\mathcal{L}(M)$ to denote the set of lines of M.

We narrow now our class of interest even further to consider the class BPav_2 of paving boolean representable simplicial complexes with dimension 2. All such simplicial complexes admit as a boolean representation a matrix containg all possible rows with exactly one entry equal to 0 (said 1-complete) and these particular boolean matrices will allow us, following a suggestion of John Rhodes (University of California at Berkeley), to introduce two notions of distance into $\mathrm{BPav}_2/\cong$ (the set of isomorphism classes of simplicial complexes lying in BPav_2). From this point on, we may use the simplicial complex $\mathcal{H}$ to refer to its isomorphism class because the following definitions do not depend on the representative. We start by defining a graph $\mathcal{D}_n$ whose vertex set consists of all elements of $\mathrm{BPav}_2/\cong$ with some fixed number $n \in \mathbb{N}$ of points. A pair of non-isomorphic simplicial complexes $\mathcal{H}$ and $\mathcal{H}'$ constitute an edge of $\mathcal{D}_n$ if and only if they admit 1-complete boolean representations being one of them obtained from the other by adding exactly one extra line. Then, we define a graph $\mathcal{D}$ whose vertex set is $\mathrm{BPav}_2/\cong$. A pair of non-isomorphic simplicial complexes $\mathcal{H} = (V, H)$ and $\mathcal{H}' = (V', H')$ constitute an edge of $\mathcal{D}$ if and only if one of the following conditions holds:

- $|V| = |V'| = n$ and $\mathcal{H}\mathcal{H}'$ is an edge of $\mathcal{D}_n$, for some $n \in \mathbb{N}$
- $|V'| = |V| + 1$ and there exist 1-complete boolean representations M (of $\mathcal{H}$) and M' (of $\mathcal{H}'$) such that

$$M' \cong \left(\begin{array}{ccc|c} & & & 1 \\ & M & & \vdots \\ & & & 1 \\ \hline 1 & \ldots & 1 & 0 \end{array}\right)$$

Using now the geodesic distance on the graphs $\mathcal{D}$ and $\mathcal{D}_n$, we are able to define metrics d (in $\mathrm{BPav}_2/\cong$) and d_n (in the set of isomorphism classes in $\mathrm{BPav}_2/\cong$ whose elements have precisely n points, for some fixed $n \in \mathbb{N}$). Observe that the previous definitions also allow the computation of distances involving the simplicial complex represented by the 1-complete boolean matrix having no lines: the uniform matroid $\mathcal{U}_{2,n}$. So, when dealing with distances, we assume that $\mathcal{U}_{2,n} \in \mathrm{BPav}_2$.

Note that the metric presented above intends to establish a distance between simplicial complexes but the objects directly involved in the explicit computation of the distance are

1-complete boolean matrices. For this reason, it becomes important to find conditions determining when are two distinct 1-complete boolean matrices representing the same simplicial complex. The original results of this monograph relate with two major questions:

1. Determine which simplicial complexes lying in BPav_2 admit alternative 1-complete boolean representations M, N with disjoint sets of lines (i.e. such that $\mathcal{L}_M \cap \mathcal{L}_N = \varnothing$)

2. Given $n \in \mathbb{N}$, get upper bounds for the distance between the uniform matroids $\mathcal{U}_{2,n}$ and $\mathcal{U}_{3,n}$

Given a boolean matrix M with columns indexed by a set V, we may define a graph $\mathscr{G}(M)$ with vertex set V and whose edges are precisely the subsets of V with 2 elements that are contained in some line of M. Interestingly, inside the class BPav_2, the set of simplexes of the simplicial complex $\mathcal{H}_M = (V, H_M)$ can be easily recovered from the set $\mathcal{L}_M$ and from the graph $\mathscr{G}(M)$ in the following way:

$$\begin{aligned} H_M &= P_{\leq 2}(V) \cup \{X \in P_3(V) \mid |X \cap L| = 2 \text{ for some line } L \in \mathcal{L}_M\} \\ &= P_{\leq 3}(V) \setminus \left(\{\text{3-anticliques of } \mathscr{G}(M)\} \cup \bigcup_{L \in \mathcal{L}_M} P_3(L) \right) \end{aligned}$$

We start by using this graph as a tool to reach an answer to a particular case of question 1: the case where all the lines of one of the matrices M or N have exactly 2 elements (said a short representation) so coinciding exactly with the set of edges of the respective graph $\mathscr{G}(M)$ or $\mathscr{G}(N)$. We conclude that there exist infinitely many such simplicial complexes (up to isomorphism) and a possible characterization of them is given by the graph associated to the short representation (say Γ). Up to isomorphism, Γ is precisely of one of the following types:

- Γ is the cyclic graph on 5 vertices
- Γ is obtained from a complete bipartite graph by removing a matching
- Γ is obtained from a complete bipartite graph by adding a trivial connected component

We proceed then to the analysis of question 1 in the remaining cases, i.e. considering that none of the representations M or N is short. From the fact that the simplicial complexes lie in the class BPav_2 we may deduce an useful property: the sets of edges of $\mathscr{G}(M)$ and $\mathscr{G}(N)$ are disjoint. So the graphs $\mathscr{G}(M)$ and $\mathscr{G}(N)$ can naturally give rise to an edge-colored graph with two colors whose edges are the elements of the set $\{\text{edges of } \mathscr{G}(M)\} \,\dot\cup\, \{\text{edges of } \mathscr{G}(N)\}$ colored according to the partition set they lie in. The concept of restriction of a simplicial complex is also useful to reduce a part of this problem to the previous case. We show that the number of vertices of such a simplicial complex lies between 5 and 9 and, establishing a forbidden configuration in the previously defined edge-colored graph, we conclude that, up to isomorphism, there are only seven simplicial complexes satisfying this desired context.

Regarding question 2, we use the concept of minimum degree (the minimum number of rows in a boolean representation) and Mantel's theorem (that presents the maximum number of edges in a triangle-free graph) to establish natural upper bounds for the distance $d_n(\mathcal{U}_{2,n}, \mathcal{U}_{3,n})$. We then improve these estimates using the characterization obtained in question 1, getting a quadratic upper bound with respect to n in both cases. Finally, we show that a natural linear upper bound is possible to attain when considering the distance $d(\mathcal{U}_{2,n}, \mathcal{U}_{3,n})$ instead.

This monograph presents the referred topics organized in 5 aditional chapters. Chapter 2 settles general definitions and notation on the main concepts: simplicial complexes, matrices, lattices, graphs and the boolean semiring. Chapter 3 introduces the notions of representation of a simplicial complex, their equivalence and the canonical representations. On Chapter 4 we focus on results about paving simplicial complexes and handle two classical examples of paving matroids. Chapter 5 presents the main original results about alternative boolean representations in BPav_2. On Chapter 6 we present Mantel's theorem and some original computations in order to get a more accurate upper bound to the distance between uniform matroids $\mathcal{U}_{2,n}$ and $\mathcal{U}_{3,n}$.

Chapter 2

General Definitions and Results

2.1 Basic Notation

Notation 2.1.1. Use $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Notation 2.1.2. Given a nonempty set V and $n \in \mathbb{N}$, define $[n] := \{m \in \mathbb{N} \mid m \leq n\}$, $P_n(V) := \{X \in 2^V \mid |X| = n\}$ and $P_{\leq n}(V) := \{X \in 2^V \mid |X| \leq n\}$. Moreover, use $v_1 v_2 \ldots v_n$ to denote V if $V = \{v_1, v_2, \ldots, v_n\}$ is a finite set.

2.2 General Concepts

2.2.1 Simplicial Complexes

Definition 2.2.1. A *simplicial complex* is a pair $\mathcal{H} = (V, H)$ where V is a finite nonempty set and $H \subseteq 2^V$ is a nonempty family of subsets of V such that $P_1(V) \subseteq H$ and H is closed under taking subsets, i.e.

$$\text{If } X \in H \text{ and } Y \subseteq X, \text{ then } Y \in H.$$

Elements of V are called *points*, elements of H are called *simplexes* and maximal elements of H with respect to set inclusion are called *facets*. Use $\operatorname{fct} \mathcal{H}$ to denote the set of facets of $\mathcal{H}$. The *dimension* of a simplex $X \in H$ is $\dim X := |X| - 1$ and the *dimension* of the simplicial complex $\mathcal{H}$ is $\dim \mathcal{H} := \max \{\dim X \mid X \in H\}$.

Definition 2.2.2. A simplicial complex $\mathcal{H} = (V, H)$ is called:

- *Simple* if $P_2(V) \subseteq H$
- *Paving* if $P_{\dim \mathcal{H}}(V) \subseteq H$
- *Uniform* if $H = P_{\leq \dim \mathcal{H}+1}(V)$
- *Pure* if all the facets have the same dimension

Definition 2.2.3. Simplicial complexes $\mathcal{H} = (V, H)$ and $\mathcal{H}' = (V', H')$ are *isomorphic*, and we write $\mathcal{H} \cong \mathcal{H}'$, if there exists a bijection $\varphi : V \to V'$ such that, for every set $X \in 2^V$, $X \in H$ if and only if $\varphi(X) \in H'$ (such a map φ is called a *simplicial complex isomorphism*).

Definition 2.2.4. A simplicial complex $\mathcal{H} = (V, H)$ is a *matroid* if it satisfies the *exchange property*:

(EP) For all $X, Y \in H$ such that $|Y| = |X| + 1$, there exists an element $y \in Y \setminus X$ such that $X \cup \{y\} \in H$

Definition 2.2.5. Given a simplicial complex $\mathcal{H} = (V, H)$ and a nonempty subset $V' \subseteq V$, the *restriction* of $\mathcal{H}$ to V' is the simplicial complex $\mathcal{H}|_{V'} := (V', H \cap 2^{V'})$.

Remark 2.2.6. Given a simplicial complex $\mathcal{H} = (V, H)$ and a subset $V' \subseteq V$, we have $\dim\ \mathcal{H}|_{V'} \leq \dim \mathcal{H}$. □

Proposition 2.2.7. *The restriction $\mathcal{H}|_{V'}$ of a paving simplicial complex $\mathcal{H} = (V, H)$ to a subset $V' \subseteq V$ is also paving.*

Proof. Let $\mathcal{H} = (V, H)$ be a paving simplicial complex and consider a set $X \in P_{\dim\ \mathcal{H}|_{V'}}(V')$. Then $X \in 2^{V'}$ and $X \in P_{\leq \dim \mathcal{H}}(V)$ (because $V' \subseteq V$ and $\dim\ \mathcal{H}|_{V'} \leq \dim \mathcal{H}$, by Remark 2.2.6); so $X \in H \cap 2^{V'}$ (because $P_{\dim \mathcal{H}}(V) \subseteq H$, i.e. H is paving). So we may conclude that $P_{\dim\ \mathcal{H}|_{V'}}(V') \subseteq H \cap 2^{V'}$ and $\mathcal{H}|_{V'}$ is paving. □

2.2.2 Lattices

Definition 2.2.8. A *(finite) partial ordered set (poset)* is a pair $(P, \leq)$ where P is a (finite) set and $\leq$ represents a partial order on P. We use P to denote $(P, \leq)$ if the partial order is clear from the context. Given elements $p, q \in P$, we say that q *covers* p if $p < q$ and, for every $x \in P$, the following condition holds:

If $p \leq x \leq q$, then $x = p$ or $x = q$.

A nonempty subset $C \subseteq P$ is a *chain* if C is totally ordered with respect to $\leq$. The *height* of a chain C is $|C| - 1$. The *height* of the poset P, $\operatorname{ht} P$ is the maximum height of a chain on P.

Definition 2.2.9. A *directed graph* is a pair $\Gamma = (V, D)$ where V is a set and $D \subseteq V \times V$. The *Hasse diagram* of a poset $(P, \leq)$ is the directed graph $\Gamma = (P, D)$ such that, for all $p, q \in P$, $(p, q) \in D$ if and only if q covers p. We adopt the usual diagram representation, whenever possible: for every $(p, q) \in D$, q is located above p and there exists a line joining the elements p and q.

Definition 2.2.10. A poset $(L, \leq)$ is a *lattice* if, for all $a, b \in L$, there exist elements:

- $a \vee_L b := \min \{x \in L \mid x \geq a, b\}$, called the *join* of a and b in L
- $a \wedge_L b := \max \{x \in L \mid x \leq a, b\}$, called the *meet* of a and b in L

$(L, \leq)$ is a *complete* lattice if, for every subset $S \subseteq L$, there exist elements:

- $\vee_L S := \min \{x \in L \mid x \geq s$ for all $s \in S\}$, called the *join* of S in L
- $\wedge_L S := \max \{x \in L \mid x \leq s$ for all $s \in S\}$, called the *meet* of S in L

Use $a \vee b, a \wedge b, \vee S, \wedge S$ to denote $a \vee_L b, a \wedge_L b, \vee_L S, \wedge_L S$ if the lattice is clear from the context.

Definition 2.2.11. Given a lattice $(L, \leq)$, we may define the *dual lattice* as $(L, \geq)$.

Remark 2.2.12. Given a lattice L, $n \in \mathbb{N}$, a finite nonempty subset $S \in P_n(L)$ and an enumeration $S = s_1 s_2 \dots s_n$, the following equalities hold: $\vee S = s_1 \vee (s_2 \vee (\dots \vee s_n))$ and $\wedge S = s_1 \wedge (s_2 \wedge (\dots \wedge s_n))$.

Proof. Given $n \in \mathbb{N}$ and a finite nonempty subset $S = s_1 s_2 \dots s_n \in P_n(L)$, we say that S satisfies the join property if $\vee S = s_1 \vee (s_2 \vee (\dots \vee s_n))$. Proceed by induction on n to show that every subset $S \in P_n(L)$ satisfies the join property, for every $n \in \mathbb{N}$, the case $n = 1$ being trivial. Suppose now that $n \geq 2$ and that every subset $S \in P_{n-1}(L)$ satisfies the join property. Consider a subset $T \in P_n(L)$ and an element $t_1 \in T$ and observe that, for every $a \in L$, $a \geq \vee(T \setminus \{t_1\})$ if and only if $a \geq t$ for every $t \in T \setminus \{t_1\}$. Therefore

$$\vee T = \min\{x \in L \mid x \geq t \text{ for every } t \in T\} = \min\{x \in L \mid x \geq t_1, \vee(T \setminus \{t_1\})\} = t_1 \vee (\vee(T \setminus \{t_1\}))$$

Now, the result follows by applying the induction hypothesis to the set $T \setminus \{t_1\} \in P_{n-1}(L)$. By duality, the second assertion holds. $\square$

Remark 2.2.13. A finite nonempty lattice admits maximum and minimum elements. Therefore every finite lattice is complete.

Proof. Let L be a finite lattice. If $L = \varnothing$ then L is trivially complete. Assume now that L is nonempty. Then, by Remark 2.2.12, there exist elements $\vee S$ and $\wedge S$ for every nonempty subset $S \subseteq L$. In particular, there exist elements $\vee L = \max L = \wedge\varnothing$ and $\wedge L = \min L = \vee\varnothing$ and so L is complete. $\square$

Note 2.2.14. Throughout this monograph, we assume that lattices are finite and nonempty.

Definition 2.2.15. Given a latice L and a subset $S \subseteq L$:

- S is a $\vee$-*subsemilattice* of L if $\min L \in S$ and $a \vee_L b \in S$ for all $a, b \in S$
- S is a $\wedge$-*subsemilattice* of L if $\max L \in S$ and $a \wedge_L b \in S$ for all $a, b \in S$
- S is a *sublattice* of L if S is a $\vee$-subsemilattice and a $\wedge$-subsemilattice of L

Remark 2.2.16. Every $\vee$-subsemilattice (or $\wedge$-subsemilattice or sublattice) is a lattice.

Proof. Let L be a lattice and $S \subseteq L$. If S is a sublattice of L, it is clear from definition that S is a lattice with $a \vee_S b = a \vee_L b$ and $a \wedge_S b = a \wedge_L b$ for all $a, b \in S$.

Suppose that S is a $\vee$-subsemilattice. Then $a \vee_S b = a \vee_L b$ and $a \wedge_S b = \vee_S \{x \in S \mid x \leq a, b\}$ for all $a, b \in S$. Note that $a \wedge_S b$ is well-defined because $\min L \in S$ and therefore $\vee_S \varnothing = \min S = \min L$. By duality, the result also holds if S is a $\wedge$-subsemilattice. $\square$

Definition 2.2.17. A lattice L is $\vee$-*generated* by a nonempty subset $A \subseteq L$ if $L = \{\vee_L X \mid X \subseteq A\}$; L is $\wedge$-*generated* by A if $L = \{\wedge_L X \mid X \subseteq A\}$.

Notation 2.2.18. Given a finite nonempty set V, a lattice L and a map $f : V \to L$, we write (L, f) if L is $\vee$-generated by $f(V)$. If $V \subseteq L$ and $f|_V$ is the identity map, we use (L, V) to denote (L, f).

Note 2.2.19. Throughout this monograph, we will only use the notation (L, A) as the lattice L is $\vee$-generated by the subset $A \subseteq L$.

Definition 2.2.20. $\vee$-generated lattices (L, A) and (L', A') are *isomorphic* if there exists a bijection $\varphi : L \to L'$ such that $\varphi(A) = A'$ and, for all $a, b \in L$, $a \leq b$ if and only if $\varphi(a) \leq \varphi(b)$.

Lemma 2.2.21. *Given a lattice (L, A) such that $\min L \notin A$ and elements $a_1, \ldots, a_k \in A$, the following conditions are equivalent:*

1. $\min L < a_1 < \vee(a_1a_2) < \ldots < \vee(a_1a_2\ldots a_k)$

2. There exists a chain $\ell_0 < \ell_1 < \ldots < \ell_k$ in L such that $a_i \leq \ell_i$ and $a_i \nleq \ell_{i-1}$, for every $i \in [k]$

Proof. Suppose that condition 1 holds and define elements $l_0 := \min L$ and $l_i = \vee(a_1a_2\ldots a_i)$ for every $i \in [k]$. We claim that $a_i \leq l_i$ and $a_i \nleq l_{i-1}$ for every $i \in [k]$. Fix some index $i \in [k]$. Clearly $a_i \leq \vee(a_1a_2\ldots a_i) = l_i$. Suppose that $a_i \leq l_{i-1} = \vee(a_1a_2\ldots a_{i-1})$. Then $\vee(a_1a_2\ldots a_i) = \vee(a_1a_2\ldots a_{i-1})$, a contradiction. So the claim and condition 2 hold.

Suppose now that condition 2 holds and that there exists $i \in [k-1]$ such that $\vee(a_1a_2\ldots a_i) = \vee(a_1a_2\ldots a_{i+1})$. Then $a_{i+1} \leq \vee(a_1a_2\ldots a_i)$. But $a_1, a_2, \ldots, a_i \leq l_i$ and so $a_{i+1} \leq \vee(a_1a_2\ldots a_i) \leq l_i$, a contradiction. So condition 1 holds. □

Definition 2.2.22. Given a lattice $(L, \leq)$, we say that a map $\varphi : L \to L$ is a *closure operator* on L if the following properties hold, for all $a, b \in L$:

- $a \leq \varphi(a)$
- If $a \leq b$ then $\varphi(a) \leq \varphi(b)$
- $\varphi(\varphi(a)) = \varphi(a)$

2.2.3 Matrices over Semirings

Definition 2.2.23. A *semiring* $(S, +, \cdot)$ consists of a set S together with binary operations $+, \cdot$ on S satisfying the following axioms:

- $(S, +)$ is a commutative monoid
- $(S, \cdot)$ is a monoid
- $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = a \cdot c + b \cdot c$, for all $a, b, c \in S$
- $a \cdot 0 = 0$ and $0 \cdot a = 0$, for every $a \in S$

Use S to denote $(S, +, \cdot)$ if the operations are implicit. $(S, +, \cdot)$ is a *commutative semiring* if $(S, \cdot)$ is a commutative monoid.

Context 2.2.24. Throughout this Subsection, let S be a semiring.

Notation 2.2.25. Given finite nonempty sets R and C, use $\mathcal{M}_{R,C}(S)$ to denote the set of all matrices with rows indexed by elements of R; columns indexed by elements of C; and entries in S. Given a matrix $M \in \mathcal{M}_{R,C}(S)$, write $M = (m_{rc})_{r\in R, c\in C}$ where $m_{rc} \in S$ is the entry of M lying in the row indexed by $r \in R$ and in the column indexed by $c \in C$, for all $r \in R$ and $c \in C$. Given subsets $R' \subseteq R$ and $C' \subseteq C$, define the *submatrix* $M[R', C'] := (m_{rc})_{r \in R', c \in C'}$ (we may use the notation $M[-, C']$ or $M[R', -]$ if $R' = R$ or $C' = C$, respectively). Given $k, \ell \in \mathbb{N}$, use $\mathcal{M}_{k,\ell}(S)$ to denote $\mathcal{M}_{[k],[\ell]}(S)$.

Note 2.2.26. There are clear bijections between R and $[|R|]$; and between C and $[|C|]$. So, throughout this monograph, we assume that $\mathcal{M}_{R,C}(S) = \mathcal{M}_{|R|,|C|}(S)$ and we use both notations in 2.2.25 to represent the same matrix, as convenient.

Definition 2.2.27. Given $k, \ell \in \mathbb{N}$, we say that matrices $M = (m_{ij}), M' = (m'_{ij}) \in \mathcal{M}_{k,\ell}(S)$ are *congruent*, and write $M \cong M'$, if there exist permutations $\sigma \in S_k$ and $\tau \in S_\ell$ such that $m'_{ij} = m_{\sigma(i),\tau(j)}$ for all $i \in [k]$ and $j \in [\ell]$.

Definition 2.2.28. Given $k \in \mathbb{N}$, we say that a square matrix $M = (m_{ij}) \in \mathcal{M}_{k,k}(S)$ is *lower unitriangular* if $m_{ii} = 1$ and $m_{ij} = 0$ for all $i, j \in [k]$ with $j > i$.

2.2.4 Graphs

Definition 2.2.29. A *(finite) graph* is a pair $\Gamma = (V, E)$ where V is a (finite) set and $E \subseteq P_2(V)$. Elements of V are called *vertices* and elements of E are called *edges*. Distinct vertices $u, v \in V$ are *adjacent* if $uv \in E$.

Definition 2.2.30. Graphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ are *isomorphic* if there exists a bijection $\varphi : V \to V'$ such that, for every $X \in E$, we have $X \in E$ if and only if $\varphi(X) \in E'$ (such a map φ is called a *graph isomorphism*).

Definition 2.2.31. Given $n, m \in \mathbb{N}$, define (up to graph isomorphism):

- The *complete graph* with n vertices as $K_n = (V, E)$ such that $|V| = n$ and $E = P_2(V)$.
- The *complete bipartite graph* with (n, m) vertices as $K_{n,m} = (V, E)$ such that $|V| = n + m$ and $E = \{ab \mid a \in A, b \in B\}$, where $V = A \dot{\cup} B$ is some partition of V such that $|A| = n$ and $|B| = m$.
- The *cyclic graph* with n vertices as $\mathcal{C}_n := (V, E)$ such that $|V| = n$ and V admits an enumeration $V = v_1 \ldots v_n$ such that $E = v_1 v_n \cup \{v_i v_{i+1} \mid i \in [n-1]\}$.

Definition 2.2.32. The *complement* of a graph $\Gamma = (V, E)$ is the graph $\Gamma^c := (V, P_2(V) \setminus E)$.

Definition 2.2.33. A *matching* with respect to some graph $\Gamma = (V, E)$ is a subset $E' \subseteq E$ of pairwise disjoint edges.

Definition 2.2.34. Given a graph $\Gamma = (V, E)$, define, for all vertices $v \in V$ and subsets $V' \subseteq V$:

- The set of *neighbours* of v as $\mathcal{N}_\Gamma(v) := \{x \in V \setminus \{v\} \mid vx \in E\}$
- The *degree* of v as $\deg_\Gamma(v) := |\mathcal{N}_\Gamma(V)|$
- The *subgraph induced* by V' as $\Gamma[V'] := (V', E \cap 2^{V'})$

Definition 2.2.35. Given a graph $\Gamma = (V, E)$ and a subset $X \subseteq V$:

- X is a *clique* of Γ if $P_2(X) \subseteq E$

- X is an *anticlique* of Γ if $P_2(X) \cap E = \varnothing$

Given $m \in \mathbb{N}$ with $m \leq n$, an m-*clique* (or m-anticlique) is a clique (or anticlique) $X \in P_m(V)$. Moreover,

- A clique X is a *superclique* of Γ if $X = (\mathcal{N}_\Gamma(u) \cup \{u\}) \cap (\mathcal{N}_\Gamma(v) \cup \{v\})$ for all distinct vertices $u, v \in X$
- An anticlique X is a *superanticlique* of Γ if $V \setminus X = \mathcal{N}_\Gamma(u) \cup \mathcal{N}_\Gamma(v)$ for all distinct vertices $u, v \in X$

Definition 2.2.36. Given a finite graph $\Gamma = (V, E)$ and $n \in \mathbb{N}_0$, a vector $(v_0, v_1, \ldots, v_n) \in V^{n+1}$ is a (closed) *path* in Γ between vertices v_0 and v_n if ($v_0 = v_n$ and) $v_{i-1}v_i \in E$ for every $i \in [n]$. A closed path $(v_0, v_1, \ldots, v_n)$ is a *cycle* if the vertices $v_1, \ldots, v_n$ are distinct. Moreover, the *length* of a given path $(v_0, v_1, \ldots, v_n) \in V^{n+1}$ is n and we write $\operatorname{len}(v_0, v_1, \ldots, v_n) = n$. Aditionaly, given distinct vertices $v, v' \in V$, let $\mathcal{P}_{v,v'}(\Gamma)$ denote the set of all paths in Γ between vertices v and v'. We say that a graph $\Gamma = (V, E)$ is *connected* if $\mathcal{P}_{v,v'} \neq \varnothing$ for all vertices $v, v' \in V$.

Remark 2.2.37. The definition of edge does not depend on the order of its elements. Therefore, given a graph $\Gamma = (V, E)$, for all distinct vertices $v_0, v_1, \ldots, v_n \in V$, $(v_0, v_1, \ldots, v_{n-1}, v_n)$ is a path in Γ if and only if $(v_n, v_{n-1}, \ldots, v_1, v_0)$ is a path in Γ.

Remark 2.2.38. Given a graph $\Gamma = (V, E)$ and vertices $v, v', v'' \in V$, if $\mathcal{P}_{v,v'}, \mathcal{P}_{v',v''} \neq \varnothing$ then $\mathcal{P}_{vv''} \neq \varnothing$.

Lemma 2.2.39. *A finite graph Γ is bipartite if and only if there are no cycles of odd length in Γ.* □

Proof. Write $\Gamma = (V, E)$. Suppose that there exists a cycle of odd length in Γ with vertex set $x_1 \ldots x_k \subseteq V$ and that Γ is a bipartite graph with partition sets $A, B \subseteq V$ with $x_1 \in A$. Then $x_2 \in B$ because $x_1x_2 \in E$. By a clear inductive generalization of this argument, $x_i \in A$ for every odd index $i \in [k]$; and $x_j \in B$ for every even index $j \in [k]$. In particular, $x_k \in A$: a contradiction because $x_1x_k \in E$.

Conversely, suppose that there are no cycles of odd length in Γ. We may assume that Γ is a connected graph, otherwise we apply the same argument to each connected component. Consider now a vertex $x \in V$, define the subset $A \subseteq V$ containing all vertices $v \in V$ such that there exists a path of even length between x and v (including the vertex x) and also the set $B := V \setminus A$. We claim that A and B are anticliques of Γ (we prove that A is an anticlique, the remaining case being analogous). To prove the claim, consider distinct vertices $a, a' \in A$ and paths $P = (x = p_0, p_1, \ldots, p_{m-1}, p_m = a)$ and $Q = (x = q_0, q_1, \ldots, q_{n-1}, q_n = a')$ of even length and suppose that $aa' \in E$. Then $(x, p_1, \ldots, p_{m-1}, a, a', q_{n-1}, \ldots, q_1, x)$ is a closed path of (odd) length $k = m + n + 1$. Therefore we may consider a closed path $R = (r_0, r_1, \ldots, r_k)$ in Γ of minimum odd length. We claim that R is a cycle. To prove the claim, suppose that R is not a cycle and consider indices $i, j \in \{0, 1, \ldots, k\}$ with $i < j$ such that $r_i = r_j$ and all the vertices $\{r_i, r_{i+1}, \ldots r_{j-1}\}$ are distinct. Now, observe that one of the closed paths $(r_i, r_{i+1}, \ldots, r_j)$ and $(r_j, r_{j+1}, \ldots, r_k = r_0, r_1, \ldots, r_i)$ has odd length, contradicting the minimality of the length of R. □

Definition 2.2.40. Given a graph $\Gamma = (V, E)$ and $k \in \mathbb{N}$, an *edge-coloring* of Γ with k colors is a map $c : E \to [k]$.

Notation 2.2.41. Given a graph $\Gamma = (V, E)$ and an edge-coloring $c : E \to [k]$ of Γ, use the pair $\mathrm{g} := (\Gamma, c)$ to denote the edge-colored graph. Given a subset $V' \subseteq V$, use the notation $\mathrm{g}\,[V'] := (\Gamma\,[V']\,, c|_{E \cap 2^{V'}})$. For all colors $i \in [k]$ and vertices $v \in V$, use the following notation:

- $\Gamma^{(i)} := (V, c^{-1}\,(i))$
- $\mathcal{N}^{(i)}(v) := \mathcal{N}_{\Gamma^{(i)}}(v)$
- $\deg^{(i)}(v) := \deg_{\Gamma^{(i)}}(v)$

Definition 2.2.42. Given $k \in \mathbb{N}$, we say that edge-coloured graphs $\mathrm{g} = (\Gamma, c)$ and $\mathrm{g}' = (\Gamma', c')$ with k colours are *isomorphic* if there exists a graph isomorphism $\varphi : V \to V'$ between the graphs Γ and Γ' and a permutation $\sigma \in S_k$ such that $\varphi(c^{-1}\,(i)) = {c'}^{-1}\,(\sigma(i))$ for every colour $i \in [k]$. Such a map φ is called an *edge-coloured graph isomorphism*. If $\mathrm{g} = \mathrm{g}'$ then φ is an *automorphism*.

2.2.5 The Boolean Semiring

Definition 2.2.43. Define the binary relation in $\mathbb{N}_0$ given by

$$\mathcal{R}_{\mathbb{B}} := \{(0,0)\} \cup \{(m,n) \mid m,n \in \mathbb{N}\}$$

Remark 2.2.44. $\mathcal{R}_{\mathbb{B}}$ is an equivalence relation. Moreover, for all $a,b,c,d \in \mathbb{N}_0$, if $(a,b),(c,d) \in \mathcal{R}_{\mathbb{B}}$, then $(a+c, b+d), (ac, bd) \in \mathcal{R}_{\mathbb{B}}$. So $\mathcal{R}_{\mathbb{B}}$ is a congruence relation and therefore $\mathbb{N}_0/\mathcal{R}_{\mathbb{B}}$ is a semiring. □

Definition 2.2.45. The *boolean semiring* is $\mathbb{B} := \mathbb{N}_0/\mathcal{R}_{\mathbb{B}}$.

Notation 2.2.46. Throughout this monograph, we use 0 and 1 to denote the respective congruence classes in $\mathbb{B}$.

Definition 2.2.47. A matrix with entries in the semiring $\mathbb{B}$ is called a *boolean matrix*.

Chapter 3

Boolean Representable Simplicial Complexes (BRSC)

3.1 Representations

Definition 3.1.1.

- Given a $\vee$-generated lattice $(L, f : Y \to L)$, we say that a subset $X \subseteq Y$ is *independent* with respect to (L, f) if there exists an enumeration $X = x_1x_2 \dots x_n$ such that $f(x_1) < \vee(f(x_1)f(x_2)) < \dots < \vee(f(x_1)f(x_2) \dots f(x_n))$.
- Given a $\vee$-generated lattice (L, A), we say that a subset $X \subseteq A$ is *independent* with respect to (L, A) if there exists an enumeration $X = x_1x_2 \dots x_n$ such that $x_1 < \vee(x_1x_2) < \dots < \vee(x_1x_2 \dots x_n)$. Define the set $H_{(L,A)}$ of all independent subsets with respect to L; and the pair $\mathcal{H}_{(L,A)} := (A, H_{(L,A)})$.

Remark 3.1.2. Given a $\vee$-generated lattice (L, A), $P_2(A) \subseteq H_{(L,A)}$.

Proof. Given distinct elements $x, y \in A$, then clearly $x \leq \vee(xy)$. If $x < \vee(xy)$, then $xy \in H_{(L,A)}$. Else, if $x = \vee(xy)$ then $y \leq x$; but x, y are distinct, so $y < x = \vee(xy)$ and we also have $xy \in H_{(L,A)}$. Hence we may conclude that $P_2(A) \subseteq H_{(L,A)}$. $\square$

Proposition 3.1.3. *$\mathcal{H}_{(L,A)}$ is a (simple) simplicial complex, for every $\vee$-generated lattice (L, A).*

Proof. $A \neq \varnothing$ by assumption and $P_2(A) \subseteq H_{(L,A)}$ by Remark 3.1.2. Now, consider sets $X \in H_{(L,A)}$ and $Y \subseteq X$. There exists an enumeration $X = x_1x_2 \dots x_n$ such that $x_1 < \vee(x_1x_2) < \dots < \vee(x_1x_2 \dots x_n)$. Also, there exist $i_1, \dots, i_k \in [n]$ with $i_1 < i_2 < \dots < i_k$ such that $Y = x_{i_1}x_{i_2} \dots x_{i_k}$. But $x_{i_1} < \vee(x_{i_1}x_{i_2}) < \dots < \vee(x_{i_1}x_{i_2} \dots x_{i_k})$ and therefore $Y \in H_{(L,A)}$. $\square$

Definition 3.1.4. A lattice (L, A) is a *lattice representation* of a simplicial complex $\mathcal{H}$ if $\mathcal{H}$ and $\mathcal{H}_{(L,A)}$ are isomorphic.

Note 3.1.5. We use Definition 3.1.4 as a notion for lattice representation of a simplicial complex to simplify notation, so we deal with simple simplicial complexes whenever we consider $\mathcal{H}_{(L,A)}$ for some $\vee$-generated lattice (L, A). A more general notion may be deduced from the first concept of independence presented in Definition 3.1.1.

Definition 3.1.6. Given a matrix $M \in \mathcal{M}_{R,V}(\mathbb{B})$, we say that a subset $X \subseteq V$ is *independent* with respect to M if there exists a subset $Y \subseteq R$ such that the matrix $M[Y, X]$ is congruent to a lower unitriangular matrix. Define the set H_M of all independent subsets with respect to M; and the pair $\mathcal{H}_M := (V, H_M)$.

Proposition 3.1.7. *Given a boolean matrix M that does not contain a column with all entries equal to 0, $\mathcal{H}_M$ is a simplicial complex.*

Proof. Write $M = (m_{rv})_{r\in R, v\in V}$. By definition, $V \neq \varnothing$. For every $v \in V$, there exists $r \in R$ such that $m_{rv} = 1$ because M does not contain a column with all entries equal to 0. Hence $P_1(V) \subseteq H_M$. Now, consider sets $X \in H_M$ and $Y \subseteq X$. Up to congruence, there exists a subset $R' \subseteq R$ such that $M' := M[R', X]$ is lower unitriangular. Write $k := |R'| = |X|$ such that $M' \in \mathcal{M}_{k,k}(\mathbb{B})$ and let $J \subseteq [k]$ denote the indices of columns also indexed by Y. Then $M'[J, J]$ is lower unitriangular and therefore $Y \in H_M$. □

Definition 3.1.8. A matrix $M \in \mathcal{M}_{R,V}(\mathbb{B})$ is a *boolean representation* of a simplicial complex $\mathcal{H}$ if $\mathcal{H}$ and $\mathcal{H}_M$ are isomorphic. Let $\mathrm{BR}(\mathcal{H})$ denote the class of all boolean representations of $\mathcal{H}$. If $\mathrm{BR}(\mathcal{H})$ is nonempty, we say that $\mathcal{H}$ is *boolean representable*.

Note 3.1.9. Throughout this monograph, given a simplicial complex $\mathcal{H} = (V, H)$ and a boolean matrix $M \in \mathrm{BR}(\mathcal{H})$, we assume (up to simplicial complex isomorphism) that V is the set indexing the columns of M and consequently $H = H_M$.

3.2 Equivalence of Representations

Most of the results presented in this Section can be found in [6, Chapters 3 and 5].

Definition 3.2.1. Define the set Mat of all boolean matrices that do not contain a column with all entries equal to 0; and the class Lat of all pairs (L, A) where L is a finite lattice $\vee$-generated by $A \subseteq L$.

Definition 3.2.2. Given a boolean matrix $M \in \mathcal{M}_{R,V}(\mathbb{B})$, define sets:

- $Z_r^M := \{v \in V \mid m_{rv} = 0\}$ for every $r \in R$; and $\mathcal{Z}^M := \{Z_r^M \mid r \in R\}$
- $Y_v^M := \cap\{Z_r^M \mid m_{rv} = 0\}$ for every $v \in V$; and $\mathcal{Y}^M := \{Y_v^M \mid v \in V\}$

Lemma 3.2.3. *Consider a boolean matrix $M = (m_{rv})_{r\in R, v\in V}$, an element $r \in R$ and a subset $S \subseteq R$. Then $m_{rv} = \sum_{s\in S} m_{sv}$ (sum in the boolean semiring $\mathbb{B}$) for every $v \in V$ if and only if $Z_r = \bigcap_{s\in S} Z_s$.*

Proof. The following statements are equivalent (sequential equivalences are clear):

1. $m_{rv} = \sum_{s\in S} m_{sv}$ for every $v \in V$
2. $m_{rv} = 0$ if and only if $m_{sv} = 0$ for every $s \in S$, for every $v \in V$
3. $v \in Z_r$ if and only if $v \in Z_s$ for every $s \in S$, for every $v \in V$
4. $Z_r = \bigcap_{s\in S} Z_s$ □

Definition 3.2.4. A boolean matrix $M = (m_{rv})_{r\in R, v\in V} \in \mathrm{Mat}$ is *closed* if it satisfies the following conditions:

C1. The rows of M are distinct

C2. The columns of M are distinct

C3. There exists $r \in R$ such that $m_{rv} = 0$ for every $v \in V$

C4. For all $s, t \in R$, there exists $r \in R$ such that $Z_r = Z_s \cap Z_t$

Let Mat_c denote the set of all closed boolean matrices.

Lemma 3.2.5. *Consider a simplicial complex $\mathcal{H} = (V, H)$, a finite nonempty set R, a matrix $M = (m_{rv})_{r\in R, v\in V} \in \mathrm{BR}(\mathcal{H})$ and a matrix N obtained from M by adding a row $(m_{sv} + m_{tv})_{v\in V}$ for some $s, t \in R$. Then $N \in \mathrm{BR}(\mathcal{H})$.*

Proof. Let α denote the vector $(m_{sv} + m_{tv})_{v\in V}$ and assume that $\alpha \notin \{\text{row vectors of } M\}$ (otherwise the result holds trivially). Let A denote the set indexing the rows of N, let $a_0 \in A$ denote the index of the row vector α, assume that $A = R \cup \{a_0\}$ and write $N = (n_{av})_{a\in A, v\in V}$. Now, clearly $H = H_M \subseteq H_N$. Conversely, let $X \in H_N$. Up to congruence, there exists $A' \subseteq A$ such that $N' := N[A', X]$ is lower unitriangular. Assume that $a_0 \in A'$ (otherwise the result holds trivially) and let α be the i^{th} row vector of N'. Let $x_0 \in X$ be the index of the i^{th} column of N' and $Y \subset X$ be the set indexing the last $|V| - i$ columns of N'. Therefore $1 = n_{a_0x_0} = m_{sx_0} + m_{tx_0}$ and so one of the entries m_{sx_0}, m_{tx_0} is equal to 1; and, for every $y \in Y$, $0 = n_{a_0y} = m_{sy} + m_{ty}$, so $m_{sy} = m_{ty} = 0$. Assume without loss of generality that $m_{sx_0} = 1$. We claim that the matrix N_s obtained from N' by replacing the row vector α by the row vector $\beta := (m_{sv})_{v\in V}$ (in the same i^{th} position) is a lower unitriangular submatrix of M. To prove the claim, observe that N' is lower unitriangular; the i^{th} entry of β is $m_{sx_0} = 1$; for every $j \in \{i+1, \dots, |V|\}$, the j^{th} entry of β is m_{sy} for some $y \in Y$, so $m_{sy} = 0$. Hence the claim holds. Moreover, the set indexing the rows of N_s is $(A' \setminus \{a_0\}) \cup \{s\} \subseteq R$ and therefore N_s is a submatrix of M. So $X \in H_M$ and we may conclude that $H_N \subseteq H_M$ and equality holds. □

Proposition 3.2.6. *Given a boolean representable simplicial complex $\mathcal{H}$, there exists a matrix in $\mathrm{BR}(\mathcal{H})$ satisfying conditions C1, C3 and C4.*

Proof. Define the map $C : \mathrm{Mat} \to \mathrm{Mat}$ such that, for every $M = (m_{rv})_{r\in R, v\in V} \in \mathcal{M}_{R,V}(\mathbb{B})$, $C(M)$ is the matrix (up to congruence) obtained from M by adding the row vectors in the set $\{(m_{sv} + m_{tv})_{v\in V} \mid s, t \in R\} \setminus \{\text{row vectors of } M\}$. Now, write $\mathcal{H} = (V, H)$ and consider a matrix

$N \in \mathrm{BR}(\mathcal{H})$. Define the matrix N_1 obtained from N by adding a row with all entries equal to 0; and the matrix N_2 obtained from N_1 by removing repeated row vectors. We claim that $H_{N_1} = H_{N_2} = H_N$. It is clear that $H_N \subseteq H_{N_1}$ and $H_{N_2} \subseteq H_{N_1}$. Conversely, consider a set $X \in H_{N_1}$. Then N_1 admits a lower unitriangular submatrix Q with columns indexed by X. But then Q does not contain repeated rows or a row with all entries equal to 0. Hence Q is also a submatrix of N and N_2 and therefore $X \in H_N, H_{N_2}$. We may then conclude that $H_{N_1} \subseteq H_N, H_{N_2}$ and so the claim holds. Hence $N_2 \in \mathrm{BR}(\mathcal{H})$ is a matrix satisfying conditions C1 and C3. Moreover, by Lemma 3.2.5, $C(N_2) \in \mathrm{BR}(\mathcal{H})$ and, inductively $C^k(N_2) \in \mathrm{BR}(\mathcal{H})$ for every $k \in \mathbb{N}$. Moreover, there exists $t \in \mathbb{N}$ such that $C^t(N_2) = C^{t+1}(N_2)$ because the number of distinct rows on a matrix with columns indexed by V is clearly limited by $2^{|V|}$. So, $C^t(N_2) \in \mathrm{BR}(\mathcal{H})$ is a matrix satisfying condition C4. Note that the map C preserves conditions C1 and C3 so the desired result holds. □

Note 3.2.7. Throughout this monograph, to simplify arguments, we assume that, given a simplicial complex $\mathcal{H}$ and a boolean representation $M \in \mathrm{BR}(\mathcal{H})$, all the rows of M are pairwise distinct. There is no loss of generality by Proposition 3.2.6.

Definition 3.2.8. Maps $\varphi : \mathrm{Mat}_c \to \mathrm{Lat}$ (or $\psi : \mathrm{Lat} \to \mathrm{Mat}_c$) are *representation-preserving* if $\mathcal{H}_M, \mathcal{H}_{\varphi(M)}$ (or $\mathcal{H}_{(L,A)}, \mathcal{H}_{\psi(L,A)}$) are pairs of isomorphic simplicial complexes, respectively, for all $M \in \mathrm{Mat}_c$ and $(L, A) \in \mathrm{Lat}$.

Notation 3.2.9.

1. Use $\mathrm{Mat}_c/\cong$ to denote the set of all congruence classes of closed boolean matrices that do not contain a column with all entries equal to 0. Given $M \in \mathrm{Mat}_c$, let $[M]_{\cong}$ denote its congruence class.
2. Use $\mathrm{Lat}/\cong$ to denote the set of all isomorphism classes of finite $\vee$-generated lattices. Given $(L, A) \in \mathrm{Lat}$, let $[(L, A)]_{\cong}$ denote its congruence class.

Context 3.2.10. We aim at defining inverse representation-preserving maps between the sets $\mathrm{Mat}_c/\cong$ and $\mathrm{Lat}/\cong$. So, up until Proposition 3.2.22, let R and V be finite nonempty sets, consider a boolean matrix $M = (m_{rv})_{r\in R, v\in V} \in \mathrm{Mat}$ and a $\vee$-generated lattice $(L, A) \in \mathrm{Lat}$.

Definition 3.2.11. Define the *lattice associated to the matrix* M, $\mathscr{L}(M)$, as the smallest $\wedge$-subsemilattice of 2^V (with respect to set inclusion) containing $\mathcal{Z}^M$.

Remark 3.2.12. $\mathscr{L}(M) = \left\{\bigcap_{s\in S} Z_s^M \mid S \subseteq R\right\}$. In particular, if M satisfies conditions C3 and C4, then $\mathscr{L}(M) = \left\{Z_r^M \mid r \in R\right\}$.

Proof. $\mathscr{L}(M)$ is a $\wedge$-subsemilattice of 2^V. Therefore $\cap\varnothing = V \in \mathscr{L}(M)$. Also, $\mathcal{Z}^M \subseteq \mathscr{L}(M)$. So $Z_r^M \cap Z_{r'}^M \in \mathscr{L}(M)$ for all $r, r' \in R$ and therefore $\bigcap_{r\in S} Z_r^M \in \mathscr{L}(M)$ for every nonempty subset $S \subseteq R$. Conversely, $\left\{\bigcap_{r\in S} Z_r^M \mid S \subseteq R\right\}$ is clearly a $\wedge$-subsemilattice of 2^V and therefore $\mathscr{L}(M) \subseteq \left\{\bigcap_{r\in S} Z_r^M \mid S \subseteq R\right\}$ by minimality.

Now, clearly $\left\{Z_r^M \mid r \in R\right\} \subseteq \left\{\bigcap_{s\in S} Z_s^M \mid S \subseteq R\right\}$. If M satisfies condition C4 (or a clear inductive generalization) then $\left\{\bigcap_{s\in S} Z_s^M \mid \varnothing \neq S \subseteq R\right\} \subseteq \{Z_r \mid r \in R\}$. If M additionally satisfies condition C3, i.e. there exists $r \in R$ such that $m_{rv} = 0$ for every $v \in V$, then $Z_r = V$ and so $V = \cap\varnothing \subseteq \{Z_r \mid r \in R\}$. Hence the last assertion holds. $\square$

Lemma 3.2.13. *$v \in Y_v^M$ for every $v \in V$.*

Proof. Let $v \in V$. If $m_{rv} = 0$ for some $r \in R$ then $v \in Z_r^M$. So we may conclude that $v \in Z_r^M$ for every $r \in R$ such that $m_{rv} = 0$, hence $v \in Y_v^M$. $\square$

Proposition 3.2.14. *$\mathscr{L}(M)$ is $\vee$-generated by $\mathcal{Y}^M$.*

Proof. Consider a set $X \in \mathscr{L}(M)$ and define $\mathcal{Y}' := \left\{Y_x^M \mid x \in X\right\}$. We claim that $X = \vee_{\mathscr{L}(M)}\mathcal{Y}'$, i.e. the following properties hold:

1. $Y_x^M \subseteq X$ for every $x \in X$
2. For every $A \in 2^V$, if $Y_x^M \subseteq A$ for every $x \in X$, then $X \subseteq A$

Let $x \in X$ and $y \in Y_x^M$. By Remark 3.2.12, there exists $S \subseteq R$ such that $X = \bigcap_{s \in S} Z_s$. So $y \in Z_r$ for every $r \in R$ such that $m_{rx} = 0$. In particular, $y \in Z_s$ for every $s \in S$ and therefore $y \in X$. So we may conclude that $Y_x^M \subseteq X$ and property 1 holds.

Now, let $A \in 2^V$ and suppose that $Y_x^M \subseteq A$ for every $x \in X$. But then, by Lemma 3.2.13, we may conclude that $X \subseteq A$ and property 2 holds. $\square$

Definition 3.2.15. Define the *matrix associated to the lattice* (L, A), $\mathscr{M}(L, A) \in \mathcal{M}_{L,A}(\mathbb{B})$, given by $\mathscr{M}(L, A) = (m_{la})_{l \in L, a \in A}$ such that $m_{la} = 0$ if and only if $l \geq a$.

Lemma 3.2.16. *Write $N := \mathscr{M}(L, A)$. The following properties hold, for all elements $x, y \in L$ and subsets $J \subseteq L$:*

1. *x is an upper bound for the set Z_x^N*
2. *$x = \vee Z_x^N$*
3. *$x \leq y$ if and only if $Z_x^N \subseteq Z_y^N$*
4. *$\bigcap_{j \in J} Z_j^N = Z_{\wedge J}^N$*

Proof. Write $N = (n_{la})_{l \in L, a \in A}$ and consider elements $x, y \in L$ and a subset $J \subseteq L$.

For every $a \in Z_x^N$, $n_{xa} = 0$ and so $x \geq a$. Hence property 1 holds.

Observe that L is $\vee$-generated by A, so there exists a subset $A' \subseteq A$ such that $x = \vee A'$. For every $b \in A'$, $x \geq b$ and so $n_{xb} = 0$, i.e. $b \in Z_x^N$. Hence we may conculde that $A' \subseteq Z_x^N$ and so $x = \vee A' \leq \vee Z_x^N$. Moreover, by property 1, $x \geq \vee Z_x^N$. Therefore, property 2 holds.

Suppose that $x \leq y$. Then y is an upper bound for the set Z_x^N by property 1. So $n_{yb} = 0$ for every $b \in Z_x^N$, i.e. $Z_x^N \subseteq Z_y^N$. Conversely, suppose that $Z_x^N \subseteq Z_y^N$. Then we get, applying property 2: $x = \vee Z_x^N \leq \vee Z_y^N = y$. Hence, property 3 holds.

Let $a \in A$. Note that $j \geq a$ for every $j \in J$ if and only if $\wedge J \geq a$, i.e. $n_{ja} = 0$ for every $j \in J$ if and only if $n_{(\wedge J)a} = 0$, i.e. $a \in \bigcap_{j \in J} Z_j$ if and only if $a \in Z_{\wedge J}^N$. So we may conclude that $\bigcap_{j \in J} Z_j = Z_{\wedge J}^N$ and property 4 holds. $\square$

Proposition 3.2.17. *$\mathscr{M}(L, A)$ is a closed matrix.*

Proof. Write $\mathscr{M}(L,A) =: N = (n_{la})_{l\in L, a\in A}$. We must prove that N verifies conditions C1-C4. By Lemma 3.2.16 (3), rows of N indexed by $x,y\in L$ are equal, i.e. $Z_x = Z_y$, if and only if $x=y$, so C1 holds. Let $a,b\in A$ and suppose that the columns indexed by a and b are equal. Then, in particular, $0 = n_{aa} = n_{ab}$ and so $a\geq b$; and $0 = n_{bb} = n_{ba}$ and so $b \geq a$. Then $a=b$ and we may conclude that C2 holds. C3 holds by considering the row indexed by $\max L$. Now, let $x,y\in L$. Then $x\wedge y\in L$ and, by Lemma 3.2.16 (4), $Z_x^N \cap Z_y^N = Z_{x\wedge y}^N$. So, C4 holds. □

Proposition 3.2.18. *The map*

$$\begin{array}{ccc} L & \to & \mathscr{L}(\mathscr{M}(L,A)) \\ x & \mapsto & Z_x^{\mathscr{M}(L,A)} \end{array}$$

is a $\vee$-generated lattice isomorphism between (L,A) and $\left(\mathscr{L}\left(\mathscr{M}(L,A)\right), \mathcal{Y}^{\mathscr{M}(L,A)}\right)$.

Proof. Write $\mathscr{M}(L,A) =: N = (n_{la})_{l\in L, a\in A}$ and let φ denote the above map.

By Lemma 3.2.16 (3), for all $x,y\in L$, then $x\leq y$ if and only if $\varphi(x) = Z_x^N \subseteq Z_y^N = \varphi(y)$. In particular, we may also conclude that φ is injective.

Recall that $\mathscr{L}(N)$ is a $\wedge$-subsemilattice of 2^A (because $N\in \mathcal{M}_{L,A}(\mathbb{B})$) and let $X\in\mathscr{L}(N)$. By Remark 3.2.12, there exists a subset $L'\subseteq L$ such that $X = \bigcap_{x\in L'} Z_x^N$. Then $\varphi(\wedge L') = Z_{\wedge L'}^N = X$ by Lemma 3.2.16 (4) and φ is surjective.

Now, for every $a\in A$, define the set $L_a := \{x\in L \mid n_{xa} = 0\}$. Then $L_a = \{x \in L \mid x\geq a\}$ and so $a = \wedge L_a$. Then the following equalities hold by using Lemma 3.2.16 (4):

$$Z_a^N = Z_{\wedge L_a}^N = \bigcap_{x\in L_a} Z_x^N = Y_a^N$$

We may then conclude that $\mathcal{Y}^N = \left\{Z_a^N \mid a\in A\right\} = \varphi(A)$. So φ is a $\vee$-generated lattice isomorphism. □

Lemma 3.2.19. *Given $v\in V$ and $r\in R$, $Y_v^M\subseteq Z_r^M$ if and only if $v\in Z_r^M$.*

Proof. Suppose first that $Y_v^M \subseteq Z_r^M$. Then $v\in Z_r^M$ by Lemma 3.2.13. Conversely, suppose that $v\in Z_r^M$. Then $m_{rv} = 0$ and so $Y_v^M = \cap\left\{Z_s^M \mid m_{sv} = 0\right\}\subseteq Z_r^M$. □

Proposition 3.2.20. *If M satisfies conditions C1, C3 and C4, then $\mathscr{M}\left(\mathscr{L}(M), \mathcal{Y}^M\right)$ and M are congruent matrices.*

Proof. By C1, all the sets Z_r^M ($r\in R$) are distinct. By an inductive generalization of C4 we get the following condition:

For every nonempty subset $S\subseteq R$, there exists $r\in R$ such that $\bigcap_{s\in S} Z_s^M = Z_r^M$.

By C3, there exists $r\in R$ such that $\wedge\,\varnothing = \max\mathscr{L}(M) = Z_r$.

Then, for every $v\in V$, there exists $r\in R$ such that $Y_v(M) = Z_r^M$ and so the set $\mathcal{Y}^M$ indexing the columns of N contains precisely $|V|$ elements. Also, by Remark 3.2.12, the set indexing the rows of N is $\mathscr{L}(M) = \left\{Z_r^M \mid r\in R\right\}$ and contains precisely $|R|$ elements. Hence we may write $\mathscr{M}(\mathscr{L}(M), \mathcal{Y}^M) =: N = \left(n_{Z_r^M Y_v^M}\right)_{r\in R, v\in V}$.

Now, the following properties hold, by definition, for all $r \in R$ and $v \in V$:

- $n_{Z_r^M Y_v^M} = 0$ if and only if $Y_v^M \subseteq Z_r^M$
- $m_{rv} = 0$ if and only if $v \in Z_r^M$

Therefore, by Lemma 3.2.19, $n_{Z_r^M Y_v^M} = 0$ if and only if $m_{rv} = 0$ for all $r \in R$ and $v \in V$. Hence M and N are congruent matrices. □

Proposition 3.2.21. *If M satisfies condition C2, then the map*

$$\begin{array}{ccc} V & \to & \mathcal{Y}^M \\ v & \mapsto & Y_v^M \end{array}$$

is a simplicial complex isomorphism between $\mathcal{H}_M$ and $\mathcal{H}_{(\mathscr{L}(M),\mathcal{Y}^M)}$.

Proof. Let H_1 and H_2 denote the set of simplexes of $\mathcal{H}_M$ and $\mathcal{H}_{(\mathscr{L}(M),\mathcal{Y}^M)}$, respectively; and let φ denote the above map. Note that φ is clearly surjective.

Consider $x, y \in V$ and suppose that $Y_x^M = Y_y^M$. Then, by Lemma 3.2.13, we have $x \in Y_y^M$ and $y \in Y_x^M$, i.e. for every $r \in R$, $m_{rx} = 0$ if and only if $m_{ry} = 0$. So the column vectors $(m_{rx})_{r\in R}$ and $(m_{ry})_{r\in R}$ are equal and therefore $x = y$. So φ is injective.

Consider now a set $X \in 2^V$.

Suppose first that $X \in H_1$. We must show that $\varphi(X) \in H_2$. As $X \in H_1$, then, up to congruence, there exists a subset $S \subseteq R$ such that $N := M[S, X]$ is lower unitriangular. Define $k := |S| = |X|$, assume that the rows and columns of N are sequentially indexed by $1, \ldots, k$, write $N = (n_{ij})_{i,j\in[k]}$ and use the notation Z_i^M and Y_j^M for all $i, j \in [k]$. Therefore the following properties hold, for every $j \in [k-1]$:

- $j \in Y_j^M$ (by Lemma 3.2.13)
- $Y_j^M \subseteq Z_p^M$ for every $p \in \{1, \ldots, j-1\}$ (because $n_{1j} = \cdots = n_{(j-1)j} = 0$)
- $j \notin Z_j^M$ (because $n_{jj} = 1$)
- $j \notin \vee\left(Y_{j+1}^M Y_{j+2}^M \ldots Y_k^M\right)$ (because Z_j^M is an upper bound for the set $Y_{j+1}^M Y_{j+2}^M \ldots Y_k^M$ and $j \notin Z_j^M$)

So we may conclude that the following strict inclusions hold:

$$Y_k^M \subset \vee\left(Y_{k-1}^M Y_k^M\right) \subset \ldots \subset \vee\left(Y_1^M Y_2^M \ldots Y_k^M\right)$$

Therefore $\varphi(X) = \left\{Y_x^M \mid x \in X\right\} \in H_2$.

Conversely, suppose that $\varphi(X) \in H_2$. We must prove that $X \in H_1$. Now, $\varphi(X) = \left\{Y_x^M \mid x \in X\right\}$ and, as $\varphi(X) \in H_2$, then there exists an enumeration $X = x_1 \ldots x_k$ such that

$$Y_{x_1}^M < \vee\left(Y_{x_1}^M Y_{x_2}^M\right) < \ldots < \vee\left(Y_{x_1}^M \ldots Y_{x_k}^M\right)$$

For every index $i \in [k]$, consider a subset $R_i \subseteq R$ such that

$$\vee\left(Y_{x_1}^M \ldots Y_{x_i}^M\right) = \bigcap_{r\in R_i} Z_r$$

(there exists such a set R_i because $\vee\left(Y^M_{x_1}\dots Y^M_{x_i}\right)$ is an element of the lattice $\mathscr{L}(M)$). Now, for every index $i \in [k]$, define the set

$$S_i := R_i \cup \ldots \cup R_k$$

and observe that

$$\bigcap_{s\in S_i} Z_s = \bigcap_{r\in R_i} Z_r$$

Moreover, it is clear that $S_k \subset S_{k-1} \subset \ldots \subset S_1$.

For every index $i \in [k]$, we claim that $x_i \in Z_s$ for every $s \in S_i$. To prove the claim, observe that $x_i \in Y^M_{x_i}$ (by Lemma 3.2.13); and so

$$x_i \in \vee\left(Y^M_{x_1}\dots Y^M_{x_i}\right) = \bigcap_{s\in S_i} Z_s$$

and the claim holds.

Now, for every index $i \in [k-1]$, we claim that there exists an element $r_i \in S_i \setminus S_{i+1}$ such that $x_{i+1} \notin Z_{r_i}$. To prove the claim, suppose that $x_{i+1} \in Z_s$ for every $s \in S_i$. Then, by Lemma 3.2.19, $Y^M_{x_{i+1}} \subseteq Z_s$ for every $s \in S_i$. Therefore

$$Y^M_{x_{i+1}} \subseteq \bigcap_{s\in S_i} Z_s = \vee\left(Y^M_{x_1}\dots Y^M_{x_i}\right)$$

a contradiction because

$$\vee\left(Y^M_{x_1}\dots Y^M_{x_i}\right) \subset \vee\left(Y^M_{x_1}\dots Y^M_{x_{i+1}}\right)$$

So the claim holds.

Finally we claim that there exists an element $r_0 \in R \setminus S_1$. To prove the claim, suppose that $S_1 = R$. Then

$$\bigcap_{r\in R} Z^M_r = \bigcap_{s\in S_1} Z^M_s = Y^M_{x_1}$$

So, for every $r \in R$, $Y^M_{x_1} \subseteq Z^M_r$ and therefore, by Lemma 3.2.19, $x_1 \in Z^M_r$ (i.e. $m_{rx_1} = 0$). This is a contradiction because $M \in \mathrm{Mat}$ does not contain a column with all entries equal to 0. Hence the claim holds.

Now, define the submatrix N of M with rows and columns sequentially indexed by $r_{k-1}, r_{k-2}, \ldots, r_1, r_0$ and $x_k, x_{k-1}, \ldots, x_2, x_1$, respectively. Write $N = (n_{ij})_{i,j\in[k]}$. We claim that N is lower unitriangular.

By the previous arguments, for every index $i \in [k]$, $n_{ii} = 1$ (because $x_{k-i+1} \notin Z_{r_{k-i}}$); and for every index $j > i$, $n_{ij} = 0$ (because $r_{k-i} \in S_{k-i} \subseteq S_{k-j+1}$ and $x_{k-j+1} \in Z_s$ for every $s \in S_{k-j+1}$). Hence the claim holds, N is lower unitriangular and therefore $X \in H_1$. □

Proposition 3.2.22. *The identity map on the set A, id_A, is a simplicial complex isomorphism between $\mathcal{H}_{(L,A)}$ and $\mathcal{H}_{\mathscr{M}(L,A)}$.*

Proof. Write $\mathcal{H}_{(L,A)} = (A, H_{(L,A)})$ and $\mathcal{H}_{\mathscr{M}(L,A)} = (A, H_{\mathscr{M}(L,A)})$ and consider a set $X \in 2^A$.

Suppose first that $X \in H_{(L,A)}$. There exists an enumeration $X = x_1x_2\ldots x_k$ such that

$x_1 < \vee(x_1x_2) < \ldots < \vee(x_1x_2\ldots x_k)$. Define $\alpha_i := \vee(x_1x_2\ldots x_i)$ for every $i \in [k]$. Observe that $x_i \not\leq \alpha_{i-1}$ for every $i \in \{2, \ldots, k\}$. So consider the submatrix N of $\mathscr{M}(L, A)$ with sequential indexing of rows and columns given by $\alpha_{k-1}, \alpha_{k-2}, \ldots, \alpha_1, \min L$ and $x_k, x_{k-1}, \ldots, x_1$, respectively. Then N is lower unitriangular and so $X \in H_{\mathscr{M}(L,A)}$.

Suppose now that $X \in H_{\mathscr{M}(L,A)}$. Up to congruence, there exists a subset $J \subseteq L$ such that $N := \mathscr{M}(L, A)[J, X]$ is lower unitriangular. Define $k := |J| = |X|$, assume that the rows and columns of N are sequentially indexed by $r_1, \ldots, r_k$ and $c_1, \ldots, c_k$, respectively and write $N = (n_{ij})_{i,j\in[k]}$.

For every $i \in [k-1]$, $c_i \not\leq \vee(c_{i+1}c_{i+2}\ldots c_k)$ because r_i is an upper bound for the set $c_{i+1}c_{i+2}\ldots c_k$ (because $n_{i(i+1)} = \ldots = n_{ik} = 0$) and $c_i \not\leq r_i$ (because $n_{ii} = 1$). Then the following strict inclusions hold:

$$c_k < \vee(c_{k-1}c_k) < \ldots < \vee(c_1c_2\ldots c_k)$$

Therefore $X \in H_{(L,A)}$. Hence id_A is a simplicial complex isomorphism. □

Corollary 3.2.23.

$$\begin{array}{rcl} \mathrm{Mat}_c/\cong & \to & \mathrm{Lat}/\cong \\ [M]_\cong & \mapsto & [\mathscr{L}(M)]_\cong \end{array} \qquad\qquad \begin{array}{rcl} \mathrm{Lat}/\cong & \to & \mathrm{Mat}_c/\cong \\ [(L,A)]_\cong & \mapsto & [\mathscr{M}(L,A)]_\cong \end{array}$$

are inverse representation-preserving maps. □

Context 3.2.24. The next two results intend to present an alternative notion of independence with respect to a boolean matrix in Mat_c.

Lemma 3.2.25. *Given a boolean matrix $M \in \mathrm{Mat}_c$ with columns indexed by a set V, a subset $X \subseteq V$ and an enumeration $X = x_1x_2\ldots x_k$, the following conditions are equivalent:*

1. $Y^M_{x_1} \subset \vee(Y^M_{x_1}Y^M_{x_2}) \subset \ldots \subset \vee(Y^M_{x_1}Y^M_{x_2}\ldots Y^M_{x_k})$

2. There exists a chain $X_0 \subset X_1 \subset \ldots \subset X_k$ in $\mathscr{L}(M)$ such that $x_i \in X_i \setminus X_{i-1}$ for every $i \in [k]$

Proof. By Lemma 2.2.21, the following conditions are equivalent:

a. $\min \mathscr{L}(M) \subset Y^M_{x_1} \subset \vee(Y^M_{x_1}Y^M_{x_2}) \subset \ldots \subset \vee(Y^M_{x_1}Y^M_{x_2}\ldots Y^M_{x_k})$

b. There exists a chain $W_0 \subset W_1 \subset \ldots \subset W_k$ in $\mathscr{L}(M)$ such that $Y_{x_i} \subseteq W_i$ but $Y_{x_i} \not\subseteq W_{i-1}$, for every $i \in [k]$

Now, conditions 1 and a are clearly equivalent because $\min \mathscr{L}(M) \notin \mathcal{Y}^M$ (i.e. M does not contain a column with all entries equal to 0 because $M \in \mathrm{Mat}$).

Moreover, using Lemma 3.2.19 and Remark 3.2.12, we may conclude that, for all $v \in V$ and $X \in \mathscr{L}(M)$, $Y^M_v \subseteq X$ if and only if $v \in X$. Therefore conditions 2 and b are equivalent by taking $X_i = W_i$ for every $i \in \{0, 1, \ldots, k\}$. □

Proposition 3.2.26. *Given a boolean matrix $M \in \mathrm{Mat}_c$ with columns indexed by a set V, a subset $X \subseteq V$ is independent with respect to M if and only if there exists an enumeration $X = x_1x_2\ldots x_k$ and a chain $X_0 \subset X_1 \subset \ldots \subset X_k$ in $\mathscr{L}(M)$ such that $x_i \in X_i \setminus X_{i-1}$ for every $i \in [k]$.*

Proof. By Lemma 3.2.25, it is enough to show that a subset $X \subseteq V$ is independent with respect to M if and only if $\{Y_x^M \mid x \in X\}$ is independent with respect to $(\mathscr{L}(M), \mathcal{Y}^M)$; and this equivalence holds by Proposition 3.2.21. □

Context 3.2.27. The last goal of this Section is to introduce a closure operator associated to the lattice $\mathscr{L}(M)$ for a given boolean matrix M.

Lemma 3.2.28. *Given a boolean matrix M with columns indexed by a set V, the map*

$$\begin{array}{ccc} 2^V & \to & 2^V \\ X & \mapsto & \cap\{Z \in \mathscr{L}(M) \mid X \subseteq Z\} \end{array}$$

is a closure operator on $(2^V, \subseteq)$.

Proof. Let φ denote the map above, write $L_0 := \mathscr{L}(M)$ and consider subsets $X, Y \in 2^V$ such that $X \subseteq Y$. Note that $X \subseteq \varphi(X)$ clearly. Moreover, recall that L_0 is a $\cap$-subsemilattice of 2^V and therefore $\varphi(X), \varphi(Y) \in L_0$. So, $\varphi(\varphi(X)) = \varphi(X)$; and $\varphi(X) \subseteq \varphi(Y)$ (because $X \subseteq Y \subseteq \varphi(Y)$). □

Definition 3.2.29. Given a boolean matrix M with columns indexed by a set V, the *closure operator* associated to the lattice $\mathscr{L}(M)$ is the map

$$\begin{array}{cccc} \mathrm{Cl}_{\mathscr{L}(M)}: & 2^V & \to & 2^V \\ & X & \mapsto & \mathrm{Cl}_{\mathscr{L}(M)}(X) := \cap\{Z \in \mathscr{L}(M) \mid X \subseteq Z\} \end{array}$$

For every set $X \in 2^V$, $\mathrm{Cl}_{\mathscr{L}(M)}(X)$ is said the *closure* of X with respect to the lattice $\mathscr{L}(M)$.

3.3 Lattice of Flats and the Canonical Boolean Representation

Most of the results presented in this Section can be found in [6, Chapter 5].

Context 3.3.1. Fix a simplicial complex $\mathcal{H}$ to simplify the statements throughout this Section. We aim at defining the lattice of flats of $\mathcal{H}$, $\mathrm{Fl}\,\mathcal{H}$ and the boolean matrix $\mathrm{Mat}\,\mathcal{H}$. Moreover, we intend to show that these are representations of $\mathcal{H}$ (as defined in Section 3.1) and that they are related (as established in Section 3.2).

Definition 3.3.2. A subset $F \subseteq V$ is a *flat* of $\mathcal{H}$ if it satisfies the following property:

$$\text{For all } X \in 2^F \cap H \text{ and } p \in V \setminus F: \quad X \cup \{p\} \in H$$

Use $\mathrm{Fl}\,\mathcal{H}$ to denote the set of all flats of $\mathcal{H}$.

Definition 3.3.3. Define the matrix $\mathrm{Mat}\,\mathcal{H} \in \mathcal{M}_{\mathrm{Fl}\,\mathcal{H}, V}(\mathbb{B})$ given by $\mathrm{Mat}\,\mathcal{H} = (m_{Fv})_{F \in \mathrm{Fl}\,\mathcal{H}, v \in V}$ such that $m_{Fv} = 0$ if and only if $v \in F$.

Lemma 3.3.4. $H_{\mathrm{Mat}\,\mathcal{H}} \subseteq H$.

Proof. We claim that $H_{\mathrm{Mat}\,\mathcal{H}} \cap P_k(V) \subseteq H$ for every $k \in \mathbb{N}$. To prove the claim, proceed by induction on k, the case $k = 0$ being trivial. Suppose now that $H_{\mathrm{Mat}\,\mathcal{H}} \cap P_{k-1}(V) \subseteq H$ for some

$k \geq 1$ and consider a set $X \in H_{\mathrm{Mat}\,\mathcal{H}} \cap P_k(V)$. Up to congruence, there exists a subset $J \subseteq \mathrm{Fl}\,\mathcal{H}$ such that $N := \mathrm{Mat}\,\mathcal{H}[J, X]$ is lower unitriangular. Let $F_1 \in \mathrm{Fl}\,\mathcal{H}$ and $v_1 \in V$ denote the index of the first row and column of N, respectively. Define $X' := X \setminus \{v_1\}$ and note that $X' \in H_{\mathrm{Mat}\,\mathcal{H}}$ because the matrix obtained from N by removing the first row and the first column is lower unitriangular. Then, by the induction hypothesis, $X' \in H$. Moreover, $X' \subseteq F_1$ (because $m_{F_1 v} = 0$ for every $v \in X'$) and $v_1 \notin F_1$ (because $m_{F_1 v_1} = 1$). So $X = X' \cup \{v_1\} \in H$ because $F_1 \in \mathrm{Fl}\,\mathcal{H}$. Hence the claim holds and so $H_{\mathrm{Mat}\,\mathcal{H}} \subseteq H$. □

Lemma 3.3.5. *Given $M \in \mathrm{BR}\,(\mathcal{H})$ and a set R indexing the rows of M, $Z_r^M \in \mathrm{Fl}\,\mathcal{H}$ for every $r \in R$.*

Proof. Write $M = (m_{rv})_{r \in R, v \in V}$. Let $r_0 \in R$ and consider $X \in 2^{Z_{r_0}^M} \cap H$ and $p \in V \setminus Z_{r_0}^M$. Up to congruence, there exists a subset $R' \subseteq R$ such that $M' := M[R', X]$ is lower unitriangular because $X \in H = H_M$. Now, $m_{r_0 p} = 1$ (because $p \notin Z_{r_0}^M$) and $m_{r_0 x} = 0$ for every $x \in X$ (because $X \subseteq Z_{r_0}^M$). Therefore, up to congruence, the matrix $M[R' \cup \{r_0\}, X \cup \{p\}]$ is lower unitriangular and so $X \cup \{p\} \in H_M = H$. Hence $Z_{r_0}^M \in \mathrm{Fl}\,\mathcal{H}$. □

Proposition 3.3.6. *If $\mathcal{H}$ is boolean representable, then every matrix in $\mathrm{BR}\,(\mathcal{H})$ is congruent to a submatrix of $\mathrm{Mat}\,\mathcal{H}$ and $\mathrm{Mat}\,\mathcal{H} \in \mathrm{BR}\,(\mathcal{H})$.*

Proof. Let $M \in \mathrm{BR}\,(\mathcal{H})$. Consider a set R indexing the rows of M and write $M = (m_{rv})_{r \in R, v \in V}$. Also, write $\mathrm{Mat}\,\mathcal{H} =: N = (n_{Fv})_{F \in \mathrm{Fl}\,\mathcal{H}, v \in V}$. By Lemma 3.3.5, $Z_r^M \in \mathrm{Fl}\,\mathcal{H}$ for every $r \in R$. Moreover, for all $r \in R$ and $v \in V$, $m_{rv} = 0$ if and only if $v \in Z_r^M$ if and only if $n_{Z_r^M v} = 0$. So M and the submatrix $\mathrm{Mat}\,\mathcal{H}[\{Z_r^M \mid r \in R\}, V]$ are congruent. So the first assertion holds.

By Lemma 3.3.4, $H_{\mathrm{Mat}\,\mathcal{H}} \subseteq H$. Consider now a simplex $X \in H$. Up to congruence, there exists a subset $R' \subseteq R$ such that $M[R', X]$ is lower unitriangular because $X \in H = H_M$. But then $\mathrm{Mat}\,\mathcal{H}[\{Z_r^M \mid r \in R'\}, X]$ is congruent to a lower unitriangular matrix. So $X \in H_{\mathrm{Mat}\,\mathcal{H}}$. Therefore we may conclude that $H \subseteq H_{\mathrm{Mat}\,\mathcal{H}}$ and equality holds. Therefore $\mathrm{Mat}\,\mathcal{H} \in \mathrm{BR}\,(\mathcal{H})$. □

Remark 3.3.7. $Z_F^{\mathrm{Mat}\,\mathcal{H}} = F$ for every $F \in \mathrm{Fl}\,\mathcal{H}$. □

Lemma 3.3.8.

1. $V \in \mathrm{Fl}\,\mathcal{H}$

2. $F \cap F' \in \mathrm{Fl}\,\mathcal{H}$ *for all* $F, F' \in \mathrm{Fl}\,\mathcal{H}$

In particular, $\mathrm{Mat}\,\mathcal{H}$ *satisfies conditions C3 and C4.*

Proof. Condition 1 holds trivially.

Consider now a set $X \in 2^{F \cap F'} \cap H$ and an element $p \in V \setminus (F \cap F')$. Note that $V \setminus (F \cap F') = (V \setminus F) \cup (V \setminus F')$. We may then consider $F_0 \in \{F, F'\}$ such that $p \in V \setminus F_0$. Then $X \in 2^{F_0} \cap H$ and therefore $X \cup \{p\} \in H$. Hence condition 2 holds.

The last assertion reformulates conditions 1 and 2 using Remark 3.3.7. □

Lemma 3.3.9. *If $P_k(V) \subseteq H$, then $P_{\leq k-1}(V) \subseteq \mathrm{Fl}\,\mathcal{H}$ for every $k \in \mathbb{N}$. In particular, $\varnothing = \min \mathrm{Fl}\,\mathcal{H}$.*

Proof. Suppose that $P_k(V) \subseteq H$ for some $k \in \mathbb{N}$ and consider a set $F \in P_{k-1}(V)$. Let $X \in 2^F \cap H$ and $p \in V \setminus F$. Then $X \cup \{p\} \in P_{\leq k}(V) \subseteq H$ and so $F \in \mathrm{Fl}\,\mathcal{H}$. □

Proposition 3.3.10. *$\mathrm{Fl}\,\mathcal{H}$ is a lattice. In particular, $\mathrm{Fl}\,\mathcal{H} = \mathscr{L}(\mathrm{Mat}\,\mathcal{H})$ is $\vee$-generated by $\mathcal{Y}^{\mathrm{Mat}\,\mathcal{H}}$.*

Proof. By property 1 and an inductive generalization of property 2 of Lemma 3.3.8, $\mathrm{Fl}\,\mathcal{H}$ is a $\wedge$-subsemilattice of 2^V, hence a lattice by Remark 2.2.16; moreover, $\mathrm{Mat}\,\mathcal{H}$ satisfies condition C4. Then by Remark 3.2.12, $\mathscr{L}(\mathrm{Mat}\,\mathcal{H}) = \left\{Z_F^{\mathrm{Mat}\,\mathcal{H}} \mid F \in \mathrm{Fl}\,\mathcal{H}\right\}$ and so $\mathscr{L}(\mathrm{Mat}\,\mathcal{H}) = \mathrm{Fl}\,\mathcal{H}$. Then $\mathrm{Fl}\,\mathcal{H}$ is $\vee$-generated by $\mathcal{Y}^{\mathrm{Mat}\,\mathcal{H}}$ by Proposition 3.2.14. □

Remark 3.3.11. If $\mathcal{H}$ is simple, then $\mathcal{Y}^{\mathrm{Mat}\,\mathcal{H}} = P_1(V)$ and $\mathrm{Mat}\,\mathcal{H}$ satisfies condition C2.

Proof. Write $\mathrm{Mat}\,\mathcal{H} = (m_{Fv})_{F \in \mathrm{Fl}\,\mathcal{H}, v \in V}$ and consider an element $v \in V$. As $P_1(V) \subseteq \mathrm{Fl}\,\mathcal{H}$, there exists a flat $F \in \mathrm{Fl}\,\mathcal{H}$ such that $Z_F^{\mathrm{Mat}\,\mathcal{H}} = \{v\}$. Then $m_{Fv} = 0$ and therefore $Y_v^{\mathrm{Mat}\,\mathcal{H}} \subseteq Z_F^{\mathrm{Mat}\,\mathcal{H}} = \{v\}$ and equality holds by Lemma 3.2.13. So $\mathcal{Y}^{\mathrm{Mat}\,\mathcal{H}} = \{\{v\} \mid v \in V\} = P_1(V)$.

The second assertion follows clearly from $P_1(V) \subseteq \mathrm{Fl}\,\mathcal{H}$. □

Proposition 3.3.12. *If $\mathcal{H}$ is simple and boolean representable, then $(\mathrm{Fl}\,\mathcal{H}, \mathcal{Y}^{\mathrm{Mat}\,\mathcal{H}})$ is a lattice representation of $\mathcal{H}$.*

Proof. By Proposition 3.3.10, $\mathrm{Fl}\,\mathcal{H} = \mathscr{L}(\mathrm{Mat}\,\mathcal{H})$. So, by Proposition 3.2.21, $\mathcal{H}_{(\mathrm{Fl}\,\mathcal{H}, \mathcal{Y}^{\mathrm{Mat}\,\mathcal{H}})}$ and $\mathcal{H}_{\mathrm{Mat}\,\mathcal{H}}$ are isomorphic simplicial complexes. And by Proposition 3.3.6, $\mathcal{H}_{\mathrm{Mat}\,\mathcal{H}} = \mathcal{H}$. □

Proposition 3.3.13. *If $\mathcal{H}$ is boolean representable then, for every subset $X \subseteq V$, $X \in H$ if and only if there exists an enumeration $X = x_1 x_2 \dots x_k$ and a chain $F_0 \subset F_1 \subset \ldots \subset F_k$ in $\mathrm{Fl}\,\mathcal{H}$ such that $x_i \in F_i \setminus F_{i-1}$ for every $i \in [k]$.*

Proof. By Proposition 3.3.6, $X \in H$ if and only if $X \in H_{\mathrm{Mat}\,\mathcal{H}}$. So, consider a subset $X \subseteq V$ and suppose first that $X \in H_{\mathrm{Mat}\,\mathcal{H}}$. Then, up to congruence, there exists a subset $G \subseteq \mathrm{Fl}\,\mathcal{H}$ such that the submatrix $N := \mathrm{Mat}\,\mathcal{H}[G, X]$ is lower unitriangular. Suppose that the rows and columns of N are sequentially indexed by $G_1, \ldots, G_k$ and $x_1, \ldots, x_k$ and define, for every index $i \in [k]$, the set $G_i' := G_1 \cap \ldots \cap G_i$. Now, $V \in \mathrm{Fl}\,\mathcal{H}$ and, for every $i \in [k]$, $G_i' \in \mathrm{Fl}\,\mathcal{H}$ (by Lemma 3.3.8). Moreover, $x_1 \in V \setminus G_1'$ and, for every $i \in \{2, \ldots, k\}$, we have $x_i \in G_{i-1}' \setminus G_i'$ by definition of the matrix $\mathrm{Mat}\,\mathcal{H}$. We get the desired result by taking the enumeration $X = x_1 \ldots x_k$ and the chain

$$G_1 \subset G_2 \subset \ldots G_k \subset V$$

Conversely, suppose now that there exists an enumeration $X = x_1 x_2 \ldots x_k$ and a chain $F_0 \subset F_1 \subset \ldots \subset F_k$ in $\mathrm{Fl}\,\mathcal{H}$ such that $x_i \in F_i \setminus F_{i-1}$ for every $i \in [k]$. Then the submatrix $\mathrm{Mat}\,\mathcal{H}[F_{k-1} F_{k-2} \ldots F_0, x_k x_{k-1} \ldots x_1]$ is lower unitriangular (if we fix the order of rows and columns) and therefore $X \in \mathcal{H}_{\mathrm{Mat}\,\mathcal{H}}$. □

Remark 3.3.14. If $\mathcal{H}$ is simple, then Proposition 3.3.13 follows directly from Proposition 3.2.26. □

Corollary 3.3.15. *If $\mathcal{H}$ is boolean representable, then $\mathrm{htFl}\,\mathcal{H} = \dim \mathcal{H} + 1$.*

Proof. We may define a map φ assigning to each chain $X_0 \subset X_1 \subset \ldots \subset X_k$ in $\mathrm{Fl}\,\mathcal{H}$ a set $x_1x_2\ldots x_k \in 2^V$ such that $x_i \in X_i \setminus X_{i-1}$. Observe that, for every $k \in \mathbb{N}$, φ maps a chain of length k to a set with k elements. Now, by Proposition 3.3.13, $\mathrm{im}\,\varphi = H$ and the result holds. □

Context 3.3.16. At this point, it is possible to define the closure operator associated to the lattice $\mathrm{Fl}\,\mathcal{H} = \mathscr{L}(\mathrm{Mat}\,\mathcal{H})$ and present some useful results involving this concept.

Definition 3.3.17. The *closure operator* associated to the lattice $\mathrm{Fl}\,\mathcal{H}$ is the map

$$\begin{array}{rccl} \mathrm{Cl}: & 2^V & \to & 2^V \\ & X & \mapsto & \mathrm{Cl}(X) := \cap\{F \in \mathrm{Fl}\,\mathcal{H} \mid X \subseteq F\} \end{array}.$$

For every set $X \in 2^V$, use the notation $\overline{X} := \mathrm{Cl}(X)$.

Proposition 3.3.18. *Given a simplex $X \in \mathrm{fct}\,\mathcal{H}$, $\overline{X} = V$.*

Proof. Suppose that there exists a flat $F \in \mathrm{Fl}\,\mathcal{H} \setminus \{V\}$ such that $X \subseteq F$. Then $V \setminus F \neq \varnothing$ and we may consider an element $p \in V \setminus F$. But then $X \in H \cap 2^F$ and so $X \cup \{p\} \in H$, contradicting the maximality of X with respect to set inclusion. □

Definition 3.3.19. Define a map

$$\begin{array}{rccl} \delta: & 2^V & \to & 2^V \\ & X & \mapsto & \delta(X) := X \cup \left\{p \in V \setminus X \mid I \cup \{p\} \notin H \text{ for some } I \in H \cap 2^X\right\} \end{array}$$

Proposition 3.3.20. *For every subset $X \in 2^V$*

$$\overline{X} = \bigcup_{n\in\mathbb{N}} \delta^n(X)$$

Proof. Write $D := \bigcup_{n\in\mathbb{N}} \delta^n(X)$. Proceed by induction on $n \in \mathbb{N}_0$ to prove that $\delta^n(X) \subseteq \overline{X}$, the case $n = 0$ being clear. We must show first the case $n = 1$, i.e., $\delta(X) \subseteq \overline{X}$. Suppose that there exists an element $p \in \delta(X) \setminus \overline{X} \subseteq \delta(X) \setminus X$. Then there exists a set $I \in H \cap 2^X \subseteq H \cap 2^{\overline{X}}$ such that $I \cup \{p\} \notin H$. This is a contradiction because $\overline{X}$ is a flat. So $\delta(X) \subseteq \overline{X}$. Consider now $n \geq 2$ and suppose that $\delta^{n-1}(X) \subseteq \overline{X}$. We must show that $\delta^n(X) \subseteq \overline{X}$. But

$$\begin{aligned} \delta^n(X) &= \delta(\delta^{n-1}(X)) \\ &\subseteq \delta(\overline{X}) && \text{(by the induction hypothesis)} \\ &\subseteq \overline{\overline{X}} = \overline{X} && \text{(by the case } n = 1\text{)} \end{aligned}$$

So we may conclude that $\delta^n(X) \subseteq \overline{X}$ for every $n \in \mathbb{N}$ and therefore $D \subseteq \overline{X}$.

Conversely, we must show that $D \in \mathrm{Fl}\,\mathcal{H}$. Consider a subset $I \in 2^D \cap H$ and an element $p \in V \setminus D$ and suppose that $I \cup \{p\} \notin H$. Observe that $X \subseteq \delta(X) \subseteq \delta^2(X) \subseteq \cdots$. So, as I is a finite set, there exists $m \in \mathbb{N}$ such that $I \subseteq \delta^m(X)$. But then $I \in H \cap 2^{\delta^m(X)}$, $p \notin \delta^m(X)$ and $I \cup \{p\} \notin H$. Hence $p \in \delta(\delta^m(X)) = \delta^{m+1}(X) \subseteq D$, a contradiction. So $I \cup \{p\} \in H$ and we may conclude that $D \in \mathrm{Fl}\,\mathcal{H}$. So $\overline{X} = D$ and the result holds. □

Lemma 3.3.21. *If $\mathcal{H}$ is a matroid, then, given a subset $X \in 2^V$, define the set $H_X \subseteq H \cap 2^X$ consisting only of the sets with maximum number of elements. Then*

$$\delta(X) = X \cup \{p \in V \setminus X \mid I \cup \{p\} \notin H \text{ for every } I \in H_X\}$$

Proof. It is clear that $X \cup \{p \in V \setminus X \mid I \cup \{p\} \notin H \text{ for every } I \in H_X\} \subseteq \delta(X)$ because $H_X \subseteq H \cap 2^X$. Conversely, consider an element $p \in \delta(X) \setminus X$. Then there exists a set $I \in H \cap 2^X$ such that $I \cup \{p\} \notin H$. If $I \subset I'$ for some $I' \in H \cap 2^X$ such that $I' \cup \{p\} \in H$ then $I \cup \{p\} \in H$. Using this argument we may assume that I is maximal in $H \cap 2^X$ with respect to set inclusion.

Consider now a set $J \in H_X$ and suppose that $J \cup \{p\} \in H$. Now, $|J| \geq |I|$ (because $J \in H_X$) and therefore $|J \cup \{p\}| > |I|$. So we may consider a simplex $J_0 \in 2^{J \cup \{p\}}$ such that $|J_0| = |I| + 1$ and apply the exchange property. There exists then an element $j \in J_0 \setminus I$ such that $I \cup \{j\} \in H$. But $I \cup \{p\} \notin H$ and therefore we may assume that $j \in J \subseteq X$. But then $I \cup \{j\} \in 2^X \cap H$ and $I \cup \{j, p\} \notin H$, contradicting the maximality of I with respect to set inclusion. Hence we may conclude that $J \cup \{p\} \notin H$ and that $\delta(X) \subseteq X \cup \{p \in V \setminus X \mid I \cup \{p\} \notin H \text{ for every } I \in H_X\}$. $\square$

Proposition 3.3.22. *If $\mathcal{H}$ is a matroid, then, for every $X \in 2^V$, $\overline{X} = \delta(X)$.*

Proof. By Proposition 3.3.20, it is clear that $\delta(X) \subseteq \overline{X}$. It remains to show that $\delta(X) \in \mathrm{Fl}\,\mathcal{H}$. Consider a subset $I \in 2^{\delta(X)} \cap H$ and an element $p \in V \setminus \delta(X)$ and suppose that $I \cup \{p\} \notin H$. If $I \subset I'$ for some $I' \in H \cap 2^{\delta(X)}$ such that $I' \cup \{p\} \in H$ then $I \cup \{p\} \in H$. Using this argument we may assume that I is maximal in $H \cap 2^{\delta(X)}$ with respect to set inclusion.

Now, we claim that $I \in H_{\delta(X)}$ (using the notation of Lemma 3.3.21). To prove the claim, suppose that $I \notin H_{\delta(X)}$ and consider $J \in H_{\delta(X)}$. Then $|J| > |I|$ and we may consider a subset $J_0 \subseteq J$ such that $|J_0| = |I| + 1$. By the exchange property, there exists an element $j \in J_0 \setminus I$ such that $I \cup \{j\} \in H$. But then $I \cup \{j\} \in 2^{\delta(X)} \cap H$ and $I \cup \{j, p\} \notin H$, contradicting the maximality of I with respect to set inclusion. Hence the claim holds.

Observe now that, as $p \in V \setminus \delta(X)$, then, applying Lemma 3.3.21, there exists a set $K \in H_X$ such that $K \cup \{p\} \in H$.

If $|I| > |K|$, then there exists a simplex $I_0 \subseteq I$ such that $|I_0| = |K| + 1$ and therefore, by the exchange property, there exists an element $i \in I_0 \setminus K$ such that $K \cup \{i\} \in H$. But, as $K \in H_X$, then the maximum number of elements of a simplex contained in X is $|K|$. Hence we may conclude that $i \notin X$. But $i \in I_0 \subseteq I \subseteq \delta(X)$ and so $i \in \delta(X) \setminus X$. Therefore, applying Lemma 3.3.21 to the element i and to the set $K \in H_X$, we get $K \cup \{i\} \notin H$, a contradiction.

On the other hand, if $|I| \leq |K|$, then $|I| < |K \cup \{i\}|$. So, there exists a simplex $K_0 \subseteq K \cup \{i\}$ such that $|K_0| = |I| + 1$. Then, by the exchange property, there exists an element $k \in K_0 \setminus I \subseteq K \setminus I$ such that $I \cup \{k\} \in H$. But then $I \cup \{k\} \in 2^{\delta(X)} \cap H$ and $I \cup \{k, p\} \notin H$, contradicting the maximality of I with respect to set inclusion.

Hence we may conclude that $I \cup \{p\} \in H$ and so $\delta(X) \in \mathrm{Fl}\,\mathcal{H}$. Hence the result holds. $\square$

Proposition 3.3.23. $\mathrm{Fl}\,\mathcal{H} = \{\overline{X} \mid X \in H\}$

Proof. It is clear that $\{\overline{X} \mid X \in H\} \subseteq \mathrm{Fl}\,\mathcal{H}$. Conversely, consider a flat $F \in \mathrm{Fl}\,\mathcal{H}$, a subset $I \in 2^F \cap H$ maximal with respect to set inclusion, and an arbitrary flat F_I containing I. We claim that $F \subseteq F_I$. To prove that claim, suppose that there exists an element $p \in F \setminus F_I \subseteq V \setminus F_I$. Then $I \cup \{p\} \in H$ because F_I is a flat. But then $I \cup \{p\} \in 2^F \cap H$, contradicting the maximality of I. So the claim holds and therefore $F = \overline{I}$. □

Proposition 3.3.24. *If $\mathcal{H}$ is boolean representable, given a subset $X \subseteq V$, then $X \in H$ if and only if there exists an enumeration $X = x_1 \dots x_k$ such that $x_i \notin \overline{x_1 \dots x_{i-1}}$ for every $i \in \{2, \dots, k\}$.*

Proof. By Proposition 3.3.13, it is enough to show that the following conditions are equivalent, for every enumeration $X = x_1 \dots x_k$:

a. There exists a chain $F_0 \subset F_1 \subset \dots \subset F_k$ in $\mathrm{Fl}\,\mathcal{H}$ such that $x_i \in F_i \setminus F_{i-1}$ for every $i \in [k]$

b. $x_i \notin \overline{x_1 \dots x_{i-1}}$ for every $i \in \{2, \dots, k\}$

Suppose that condition a holds. Then, for every index $i \in \{2, \dots, k\}$, $x_1, \dots, x_{i-1} \in F_{i-1}$ and therefore $\overline{x_1 \dots x_{i-1}} \subseteq F_{i-1}$; so $x_i \notin \overline{x_1 \dots x_{i-1}}$. Conversely, if condition b holds, then condition a also holds by taking $F_0 := \varnothing$ and $F_i := \overline{x_1 \dots x_i}$ for every $i \in [k]$. □

3.4 Definitions and Results on BRSC

Most of the results presented in this Section can be found in [6, Chapter 5].

Definition 3.4.1. A boolean matrix is said to be *1-complete* if it contains all possible rows with exactly one entry equal to 0. Given a simplicial complex $\mathcal{H}$, use $\mathrm{BR}_{\text{1-complete}}(\mathcal{H})$ to denote the set of 1-complete boolean representations of $\mathcal{H}$.

Definition 3.4.2. Given finite nonempty sets R, V and a boolean matrix $M \in \mathcal{M}_{R,V}(\mathbb{B})$, the set of *lines* of M is $\mathcal{L}_M := \{Z_r^M \mid r \in R, 2 \leq |Z_r^M| \leq |V| - 1\}$ (recall the notation introduced in Definition 3.2.2). The *graph associated to the matrix* M is $\mathscr{G}(M) = (V, E)$ given by $E := \bigcup_{L \in \mathcal{L}_M} P_2(L)$.

Definition 3.4.3. A boolean matrix M is short if $|L| = 2$ for every line $L \in \mathcal{L}_M$. Given a simplicial complex $\mathcal{H}$, use $\mathrm{BR}_{\text{short}}(\mathcal{H})$ to denote the set of short boolean representations of $\mathcal{H}$.

Note 3.4.4. Some results of this monograph depend on the set of lines of a given boolean matrix being a partial euclidean geometry, so we may define this concept.

Definition 3.4.5. Given a finite nonempty set V and a family $\mathcal{L} \subseteq 2^V$, we say that $\mathcal{L}$ is a *partial euclidean geometry* (PEG) if the following conditions hold:

- $|L| \geq 2$, for every $L \in \mathcal{L}$
- $|L \cap L'| \leq 1$, for all distinct $L, L' \in \mathcal{L}$

Proposition 3.4.6. *Given a simplicial complex $\mathcal{H}$, the following conditions are equivalent:*

1. $\mathcal{H}$ is boolean representable

2. For every simplex $X \in H \setminus \{\varnothing\}$, there exists an element $x \in X$ such that $x \notin \overline{X \setminus \{x\}}$

Proof. Write $\mathcal{H} = (V, H)$. Suppose that $\mathcal{H}$ is boolean representable and consider a simplex $X \in H \setminus \{\varnothing\}$. Then, by Proposition 3.3.24, there exists an enumeration $X = x_1 \dots x_{k-1}$ such that $x_k \notin \overline{x_1 \dots x_k}$ and so condition 2 holds.

Conversely, suppose that condition 2 holds, consider $k \in [\dim \mathcal{H} + 1]$ and a simplex $X \in H \cap P_k(V)$. Write $X_k := X$ and consider an element $x_k \in X_k$ such that $x_k \notin \overline{X_k \setminus \{x_k\}}$. Now, recursively, for every $i \in [k-1]$, define $X_{k-i} := X_{k-i+1} \setminus \{x_{k-i+1}\}$ and consider an element $x_{k-i} \in X_{k-i}$ such that $x_{k-i} \notin \overline{X_{k-i} \setminus \{x_{k-i}\}}$. For every $i \in \{0, 1, \dots, k\}$ there exist such elements x_{k-i} by condition 2, because $X_{k-i} \subseteq X \in H$ and therefore $X_{k-i} \in H \setminus \{\varnothing\}$. Then $\varnothing \subset X_1 \subset X_2 \subset \dots \subset X_k$. Hence $\varnothing \subseteq \overline{X_1} \subseteq \overline{X_2} \subseteq \dots \subseteq \overline{X_k}$. Moreover, for all indices $p, q \in [k]$, with $q \geq p$, $x_p \in X_p$, so $x_p \in X_q$ and therefore $x_p \in \overline{X_q}$; but $x_p \notin \overline{X_{p-1}}$. Hence, we may consider the submatrix M of $\operatorname{Mat} \mathcal{H}$ such that the rows and columns of M are sequentially indexed by $\overline{\varnothing}, \overline{X_1}, \overline{X_2}, \dots, \overline{X_{k-1}}$ and $x_1, x_2, \dots, x_k$, respectively. By the previous argument, M is lower unitriangular and therefore $X \in H_{\operatorname{Mat} \mathcal{H}}$ and we may conclude that $H \subseteq H_{\operatorname{Mat} \mathcal{H}}$. Moreover, by Lemma 3.3.4, $H_{\operatorname{Mat} \mathcal{H}} \subseteq H$ and equality holds. So $\operatorname{Mat} \mathcal{H} \in \operatorname{BR}(\mathcal{H})$ and therefore condition 1 holds. □

Lemma 3.4.7. *Every matroid is pure.*

Proof. Consider a simplicial complex $\mathcal{H} = (V, H)$ satisfying the exchange property. Note that $\operatorname{fct} \mathcal{H} \neq \varnothing$ because H is nonempty and finite so there exist maximal elements in H with respect to set inclusion. Suppose now that there exist distinct sets $X, Y \in \operatorname{fct} \mathcal{H}$ with $k := |X| < |Y|$ and consider $Y' \in P_{k+1}(Y)$. Observe that $P_{k+1}(Y) \subseteq 2^Y \subseteq H$ because $Y \in H$ and $\mathcal{H}$ is a simplicial complex. So $Y' \in H$. There exists an element $y_1 \in Y' \setminus X$ such that $X_1 := X \cup \{y_1\} \in H$ because $|Y'| = k + 1 = |X| + 1$ and $\mathcal{H}$ is a matroid. But then $X \subset X_1 \in H$, contradicting X being maximal with respect to set inclusion. □

Proposition 3.4.8. *Every matroid is boolean representable.*

Proof. Consider a matroid $\mathcal{H} = (V, H)$, a simplex $X \in H \setminus \{\varnothing\}$, a point $x \in X$ and write $X_x := X \setminus \{x\}$. By Proposition 3.4.6, it is enough to show that $x \notin \overline{X_x}$. Consider now a subset $I \in H \cap 2^{X_x}$. Then $I \cup \{x\} \subseteq X \in H$ and therefore $I \cup \{x\} \in H$. Moreover, by Proposition 3.3.22,

$$\overline{X_x} = X_x \cup \left\{p \in V \setminus X_x \mid I \cup \{p\} \notin H \text{ for some } I \in H \cap 2^{X_x}\right\}$$

So we may conclude that $x \notin \overline{X_x}$ and the result holds. □

Remark 3.4.9. There exist boolean representable simplicial complexes that are not matroids. An example is provided in Section 3.5.

Proposition 3.4.10. *The restriction of a boolean representable simplicial complex $\mathcal{H} = (V, H)$ to a subset $V' \subseteq V$ is also boolean representable. In particular, if $M \in \operatorname{BR}(\mathcal{H})$ then $M[-, V'] \in \operatorname{BR}(\mathcal{H}|_{V'})$.*

Proof. Write $M' := M[-, V']$, consider a set R indexing the rows of M and M' and recall that $\mathcal{H}|_{V'} = (V', H \cap 2^{V'})$. Observe that $H = H_M$ because $M \in \operatorname{BR}(\mathcal{H})$. Then we aim at proving that $H_{M'} = H_M \cap 2^{V'}$. Clearly $H_{M'} \subseteq H_M \cap 2^{V'}$. Conversely, consider a set $X \in H_M \cap 2^{V'}$. Up to congruence, there exists a subset $S \subseteq R$ such that $M[S,X]$ is lower unitriangular. But $M[S,X] = M'[S,X]$ because $X \in 2^{V'}$. Therefore $X \in H_{M'}$ and we may conclude that $H_M \cap 2^{V'} \subseteq H_{M'}$ and equality holds. □

Remark 3.4.11. Consider a boolean representable simplicial complex $\mathcal{H} = (V, H)$, a matrix $M \in \mathrm{BR}(\mathcal{H})$ and a subset $V' \subseteq V$. Then, for every line $L' \in \mathcal{L}_{M[-,V']}$ there exists a line $L \in \mathcal{L}_M$ such that $L' \subseteq L$.

Proof. Consider a line $L' \in \mathcal{L}_{M[-,V']}$. Clearly, there exists $F \in \mathcal{L}_M \cup \{V\}$ such that $L' \subseteq F$. Observe that $|L'| \leq |V'| - 1$ because $L' \in \mathcal{L}_{M[-,V']}$. Therefore, $|F| \leq |L'| + (|V| - |V'|) \leq |V| - 1$. So $F \in \mathcal{L}_M$. □

3.5 Examples

Example 3.5.1. Consider sets $V := [4]$ and $H := P_{\leq 2}(V) \cup \{123, 124\}$. Then $\mathcal{H} := (V, H)$ is a simplicial complex; $\dim \mathcal{H} = 2$; $\mathcal{H}$ is simple and paving; $\mathrm{fct}\, \mathcal{H} = \{34, 123, 124\}$ and $\mathrm{Fl}\, \mathcal{H} = P_{\leq 1}(V) \cup \{12, V\}$. Moreover, $\mathcal{H}$ is boolean representable but not a matroid.

Proof. Start by observing that V is a finite nonempty set and H is a nonempty subset of 2^V such that $P_1(V) \subseteq H$ and closed under taking subsets. Then $\mathcal{H}$ is a simplicial complex. Recall that $\dim X := |X| - 1$ for every simplex $X \in H$; and $\dim \mathcal{H} := \max\{\dim X \mid X \in H\}$. So $\dim \mathcal{H} = 2$. The maximal elements of H with respect to set inclusion are $\mathrm{fct}\, \mathcal{H} = \{34, 123, 124\}$. Observe that $P_2(V) \subseteq H$. So, $\mathcal{H}$ is clearly simple; moreover, $\mathcal{H}$ is also paving because $\dim \mathcal{H} = 2$.

Recall that, by Proposition 3.3.23, $\mathrm{Fl}\, \mathcal{H} = \{\overline{X} \mid X \in H\}$. Moreover, $V \in \mathrm{Fl}\, \mathcal{H}$ (by Lemma 3.3.8); $P_{\leq 1}(V) \subseteq \mathrm{Fl}\, \mathcal{H}$ because $P_2(V) \subseteq H$ (by Lemma 3.3.9); and $\overline{X} = V$ for every $X \in \mathrm{fct}\, \mathcal{H}$ (by Proposition 3.3.18). Hence it suffices to compute the closure of the elements of the set $H \setminus \{P_{\leq 1}(V) \cup \mathrm{fct}\, \mathcal{H}\} = \{12, 13, 14, 23, 24\}$. We claim first that $12 \in \mathrm{Fl}\, \mathcal{H}$. To prove the claim, consider a subset $I \in 2^{12} \cap H = \{\varnothing, 1, 2, 12\}$ and an element $p \in V \setminus 12 = 34$. We must show that $I \cup \{p\} \in H$. Observe that, if $|I| \leq 1$ then $I \cup \{p\} \in P_{\leq 2}(V) \subseteq H$. Assume then that $I = 12$ and $p \in 34$ and observe that $123, 124 \in H$ so the claim holds. Therefore $12 \in \mathrm{Fl}\, \mathcal{H}$ and so $\overline{12} = 12$. Now, we claim that $\overline{13} = V$. To prove the claim, use Proposition 3.3.20: observe that $134 \notin H$ and so $4 \in \delta(13) \subseteq \overline{13}$; hence $34 \subseteq \overline{13}$. So, by the properties of the closure operator, $\overline{34} \subseteq \overline{\overline{13}} = \overline{13}$. But $\overline{34} = V$ (because $34 \in \mathrm{fct}\, \mathcal{H}$) and therefore $\overline{13} = V$. Similarly, we may conclude that $\overline{14} = \overline{23} = \overline{24} = V$ and therefore $\mathrm{Fl}\, \mathcal{H} = P_{\leq 1}(V) \cup \{12, V\}$.

Consider the matrix

$$M := \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

We claim that $M \in \mathrm{BR}(\mathcal{H})$. Observe that, if we fix the order of rows and columns, the submatrices $M[r_1r_3r_2, 312]$, $M[r_1r_3r_2, 412]$ and $M[r_2r_1, 43]$ are lower unitriangular and so $123, 124, 34 \in H_M$. Moreover, $134, 234 \notin H_M$ because all the rows of the submatrices $M[-, 134]$ and $M[-, 234]$ have at most one entry equal to 0. Moreover, recall that the set H_M is closed under takins subsets. Therefore $H_M = P_{\leq 2}(V) \cup \{123, 124\} = H$ and so $M \in \mathrm{BR}(\mathcal{H})$.

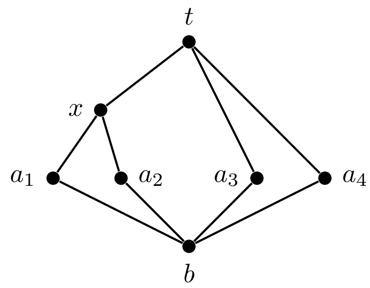
We are also able to construct the matrix $\operatorname{Mat}\mathcal{H}$ and, according to Proposition 3.3.6, M is congruent to a submatrix of $\operatorname{Mat}\mathcal{H}$.

$$
\operatorname{Mat}\mathcal{H} := \begin{array}{c} \\ \varnothing \\ 1 \\ 2 \\ 3 \\ 4 \\ 12 \\ V \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}
$$

M is in fact a submatrix of $\operatorname{Mat}\mathcal{H}$ if we consider $r_1 = 12$, $r_2 = 3$ and $r_3 = 2$.

Finally, we must prove that $\mathcal{H}$ does not satisfy the exchange property so it is not a matroid. For, observe that $34, 123 \in H$ but $134, 234 \notin H$. $\square$

Example 3.5.2. Consider the lattice L determined by the Hasse diagram



and define the subset $A := \{a_i \mid i \in [4]\}$. Then L is $\vee$-generated by A and $\mathcal{H}_{(L,A)} \cong \mathcal{H}$ (considering the simplicial complex $\mathcal{H}$ from Example 3.5.1).

Proof. L is $\vee$-generated by A because $x = \vee a_1a_2$, $t = \vee a_1a_3$ and $b = \vee\, \varnothing$. Moreover, as mentioned in Remark 3.1.2, according to the adopted definition, the simplicial complex $\mathcal{H}_{(L,A)}$ is simple, i.e. $P_2(V) \subseteq H_{(L,A)}$. Moreover, $\operatorname{ht} L = 3$. Now, for every index $i \in \{3, 4\}$

$$
\underbrace{(\vee a_1)}_{a_1} < \underbrace{(\vee a_1a_2)}_{x} < \underbrace{(\vee a_1a_2a_i)}_{t}
$$

and therefore $a_1a_2a_i \in H_{(L,A)}$. On the other hand, for every index $i \in \{1, 2\}$, $a_ia_3a_4 \notin H_{(L,A)}$ because $\vee a_ia_3 = \vee a_ia_4 = \vee a_3a_4 = t$ and therefore the length of every chain contained in $a_ia_3a_4$ is at most 2.

Use $\mathcal{H} = (V, H)$ to denote the simplicial complex of the example 3.5.1 and note that $\mathcal{H} \cong \mathcal{H}_{(L,A)}$ (through the bijection $\varphi : V \to A$ such that $i \mapsto a_i$ for every $i \in [4]$). Therefore, (L, A) is a lattice representation of $\mathcal{H}$. In particular, observe that $\mathrm{Fl}\,\mathcal{H}$ is $\vee$-generated by V and therefore (L, A) and $(\mathrm{Fl}\,\mathcal{H}, V)$ are isomorphic lattices (through the same bijection). $\square$

Definition 3.5.3. Given $n, m \in \mathbb{N}$ with $n \geq m$, let $\mathcal{U}_{m,n} = (V, H)$ denote the uniform simplicial complex (up to isomorphism) such that $|V| = n$ and $H = P_{\leq m}(V)$.

Note 3.5.4. Throughout this monograph, given $n, m \in \mathbb{N}$ such that $n \geq m$, we assume that the set of points of the simplicial complex $\mathcal{U}_{m,n}$ is $[n]$.

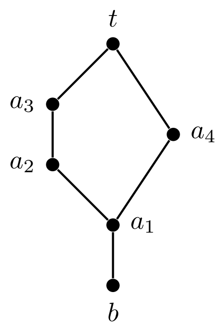
Remark 3.5.5. Given $n, m \in \mathbb{N}$ such that $n \geq m$, the simplicial complex $\mathcal{U}_{m,n}$ is a matroid. $\square$

Lemma 3.5.6. *Given $n, m \in \mathbb{N}$ such that $n \geq m$, $\mathrm{Fl}\,\mathcal{U}_{m,n} = P_{\leq m-1}([n]) \cup \{[n]\}$.*

Proof. By Lemma 3.3.9, $P_{\leq m-1}([n]) \subseteq \mathrm{Fl}\,\mathcal{H}$ and therefore, for every $X \in P_{\leq m-1}([n])$, $\overline{X} = X$. Moreover, every subset $X \in P_m(V)$ is a facet of this simplicial complex and therefore, by Proposition 3.3.18, $\overline{X} = [n]$. So, applying Proposition 3.3.23, we get

$$\mathrm{Fl}\,\mathcal{U}_{m,n} = \left\{\overline{X} \mid X \in P_{\leq m}([n])\right\} = P_{\leq m-1}([n]) \cup \{[n]\}$$ $\square$

Example 3.5.7. Consider the lattice L determined by the Hasse diagram



and define the subset $A := \{a_i \mid i \in [4]\}$. Then L is $\vee$-generated by A and $H_{(L,A)} \cong \mathcal{U}_{4,4}$.

Proof. L is $\vee$-generated by A because $t = \vee a_3 a_4$ and $b = \vee\, \varnothing$. Moreover, $\mathrm{ht}\, L = 4$. Now, observe that

$$\underbrace{(\vee a_1)}_{a_1} < \underbrace{(\vee a_1 a_2)}_{a_2} < \underbrace{(\vee a_1 a_2 a_3)}_{a_3} < \underbrace{(\vee a_1 a_2 a_3 a_4)}_{t}$$

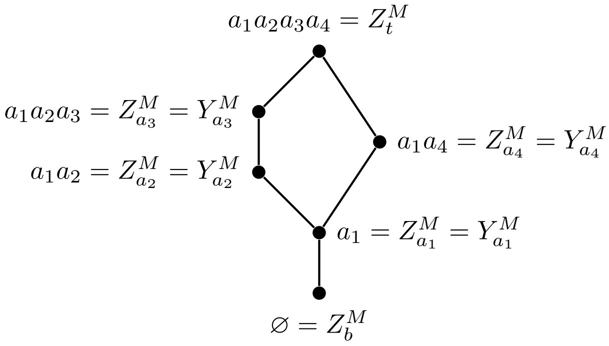
and therefore $A \in H_{(L,A)}$. So, $H_{(L,A)} = P_{\leq 3}(A)$ because the set $H_{(L,A)}$ is closed under taking subsets. So $\mathcal{H}_{(L,A)} \cong \mathcal{U}_{3,4}$. $\square$

Example 3.5.8. Consider the $\vee$-generated lattice (L, A) from Example 3.5.7. Then

$$\mathscr{M}(L, A) = \begin{array}{c} \\ b \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ t \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

and $\left(\mathscr{L}(\mathscr{M}(L,A)),\mathcal{Y}^{\mathscr{M}(L,A)}\right)\cong(L,A)$.

Proof. Write $\mathscr{M}(L,A)=:M:=(m_{la})_{l\in L,a\in A}$. To determine the lattice $\mathscr{L}(M)$ and the set of generators $\mathcal{Y}^M$ we must compute the sets $Z_l^M:=\{a\in A\mid m_{la}=0\}$ and $Y_a^M:=\cap\left\{Z_l^M\mid m_{la}=0\right\}$, for all $l\in L$ and $a\in A$. So, the lattice $\mathscr{L}(M)$ is determined by the following Hasse diagram:



The lattices (L,A) and $\left(\mathscr{L}(M),\mathcal{Y}^M\right)$ are isomorphic through the bijection $A\to\mathcal{Y}^M$ such that $a\mapsto Y_a^M$. □

Chapter 4

Paving Boolean Representable Simplicial Complexes

4.1 Definitions and Results on paving BRSC

Most of the results presented in this Section can be found in [6, Chapter 6].

Notation 4.1.1. Given $n, m \in \mathbb{N}$ with $m \geq n$, use Pav_n (or $\mathrm{Pav}_n^{(m)}$) to denote the class of all paving simplicial complexes of dimension n (with m vertices); and use BPav_n (or $\mathrm{BPav}_n^{(m)}$) to denote the class of all boolean representable paving simplicial complexes of dimension n (with m vertices).

Proposition 4.1.2. *Given $n \in \mathbb{N}$, a simplicial complex $\mathcal{H} = (V, H) \in \mathrm{Pav}_n$ and distinct flats $F, F' \in \mathrm{Fl}\,\mathcal{H} \setminus \{V\}$, we have $|F \cap F'| \leq n - 1$.*

Proof. Define $F_0 := F \cap F'$, suppose that $|F_0| \geq n$ and consider distinct elements $x_1, \ldots, x_n \in F_0$. Observe that $P_n(V) \subseteq H$ (because $\mathcal{H}$ is paving) and so, by Lemma 3.3.9, $P_{\leq n-1}(V) \subseteq \mathrm{Fl}\,\mathcal{H}$. We may then define flats $F_i := x_1 \ldots x_i$ for every index $[n-1]$. Moreover, we may assume that $F_0 \neq F$ (because F, F' are distinct flats) and consider an element $f \in F \setminus F_0$. Hence the submatrix of $\mathrm{Mat}\,\mathcal{H}$ with rows sequentially indexed by flats $F_0, F_{n-1}, F_{n-2}, \ldots, F_2, F_1, \varnothing$ and columns sequentially indexed by elements $f, x_n, x_{n-1}, \ldots, x_2, x_1$ is lower unitriangular. Hence $x_1 \ldots x_n \cup f \in 2^F \cap H_{\mathrm{Mat}\,\mathcal{H}}$ and therefore, by Lemma 3.3.4, $x_1 \ldots x_n f \in 2^F \cap H$. But then, for every element $p \in V \setminus F$, $x_1 \ldots x_n f p \in H$, a contradiction because $\dim \mathcal{H} = n$. $\square$

Remark 4.1.3. Given a simplicial complex $\mathcal{H} = (V, H) \in \mathrm{BPav}_2$, a matrix $M \in \mathrm{BR}(\mathcal{H})$ and distinct lines $L, L' \in \mathcal{L}_M$, we have $|L \cap L'| \leq 1$. $\square$

4.1.1 Dimension 2

Remark 4.1.4. Given a 1-complete boolean matrix M with columns indexed by a set V, we have $P_{\leq 2}(V) \subseteq H_M$. In particular, if $\mathcal{L}_M = \varnothing$ then $\mathcal{H}_M \cong \mathcal{U}_{2,n}$.

Proof. Let R be the set indexing the rows of M and write $M = (m_{rv})_{r\in R, v\in V}$. Consider a subset $X \in P_2(V)$ and write $X = xy$. There exist distinct $r, s \in R$ such that $Z_r^M = \{x\}$ and $Z_s^M = \{y\}$ because M is 1-complete; and therefore $m_{sx} = m_{ry} = 1$ and $m_{rx} = m_{sy} = 0$. Hence, up to congruence, the submatrix $M[rs, xy]$ is lower unitriangular. Therefore, $P_2(V) \subseteq H_M$ and so $P_{\leq 2}(V) \subseteq H_M$.

Suppose now that $\mathcal{L}_M = \varnothing$ and there exists a subset $X \in P_3(V) \cap H_M$. Then there exists a subset $R' \subseteq R$ such that $M[R', X]$ is congruent to a lower unitriangular matrix. But then there exists an element $r \in R$ such that $2 \leq \left|Z_r^M\right| \leq |V| - 1$, a contradiction, because $\mathcal{L}_M = \varnothing$. Therefore, if $\mathcal{L}_M = \varnothing$ then $H_M = P_{\leq 2}(V)$ and so $\mathcal{H}_M \cong \mathcal{U}_{2,n}$. □

Proposition 4.1.5. *The following properties hold:*

1. *Given a simplicial complex $\mathcal{H} \in \mathrm{BPav}_2$ and a matrix $M \in \mathrm{BR}(\mathcal{H})$, then $\mathcal{L}_M$ is a nonempty PEG.*
2. *If M is a 1-complete boolean matrix such that $\mathcal{L}_M$ is a nonempty PEG, then $\mathcal{H}_M \in \mathrm{BPav}_2$.*

Proof. Consider a simplicial complex $\mathcal{H} \in \mathrm{BPav}_2$ and a matrix $M \in \mathrm{BR}(\mathcal{H})$. Let R be the set indexing the rows of M and write $M = (m_{rv})_{r\in R, v\in V}$. Then, by Remark 4.1.4, $\mathcal{L}_M \neq \varnothing$. Recall that $Z_r^M \in \mathrm{Fl}\,\mathcal{H}$ for every $r \in R$ by Lemma 3.3.5. Then, by Proposition 4.1.2, $\left|Z_r^M \cap Z_s^M\right| \leq 1$ for all distinct $r, s \in R$ and so $\mathcal{L}_M$ is a PEG. Hence property 1 holds.

Consider now a 1-complete boolean matrix M such that $\mathcal{L}_M$ is a nonempty PEG. By Remark 4.1.4, it remains to show that $P_3(V) \cap H_M \neq \varnothing$ and that $P_4(V) \cap H_M = \varnothing$. Consider then a line $L \in \mathcal{L}_M$, distinct elements $p, q \in L$, an element $x \in V \setminus L$ and an element $r \in R$ such that $L = Z_r^M$. There exist distinct elements $s_1, s_2 \in R$ such that $Z_{s_1}^M = \{p\}$ and $Z_{s_2}^M = \{q\}$ because M is

1-complete. Then $m_{rx} = 1$, $m_{rp} = m_{rq} = 0$; $m_{s_2p} = 1$, $m_{s_2q} = 0$; and $m_{s_1q} = 1$. Hence the submatrix $M[rs_2s_1, xpq]$ is lower unitriangular (if we fix the order of rows and columns) and therefore $xpq \in P_3(V) \cap H_M$.

On the other hand, suppose that there exist a set $X \in P_4(V) \cap H_M$. Then there exist a subset $R' \subseteq R$ such that $M[R', X]$ is congruent to a lower unitriangular matrix. But then there exist distinct elements $r, s \in R$ such that $Z_r^M, Z_s^M \in \mathcal{L}_M$ and $\left|Z_r^M \cap Z_s^M\right| \geq 2$, a contradiction, because $\mathcal{L}_M$ is a PEG. Hence property 2 holds. □

Proposition 4.1.6. *Given a simplicial complex $\mathcal{H} = (V, H) \in \mathrm{BPav}_2$ and a matrix $M \in \mathrm{BR}(\mathcal{H})$:*

$$\begin{aligned} H = H_M &= P_{\leq 2}(V) \cup \bigcup_{L\in\mathcal{L}_M} \{X \in P_3(V) \mid |X \cap L| = 2\} \\ &= P_{\leq 3}(V) \setminus \left(\{\textit{3-anticliques of } \mathscr{G}(M)\} \cup \bigcup_{L\in\mathcal{L}_M} P_3(L) \right) \end{aligned}$$

Proof. Consider a set R indexing the rows of M, write $M = (m_{rv})_{r\in R, v\in V}$, define $H' := H \cap P_3(V)$ and sets

$$U_1 := \bigcup_{L\in\mathcal{L}_M} \{X \in P_3(V) \mid |X \cap L| = 2\}$$

$$U_2 := P_3(V) \setminus \left(\{\text{3-anticliques of } \mathscr{G}(M)\} \cup \bigcup_{L \in \mathcal{L}_M} P_3(L) \right)$$

We claim that $H' = U_1 = U_2$. To prove the claim consider first a set $Y \in H'$. Up to congruence, there exists a subset $R' \subseteq R$ such that the submatrix $M' := M[R', Y]$ is lower unitriangular because $Y \in H = H_M$. Suppose that the rows and columns of M' are sequentially indexed by r_1, r_2, r_3 and y_1, y_2, y_3, respectively. Then $m_{r_1 y_1} = 1$ and $m_{r_1 y_2} = m_{r_1 y_3} = 0$ and therefore $2 \leq \left|Z^M_{r_1}\right| \leq |V| - 1$. So $Z^M_{r_1} \in \mathcal{L}_M$ and $\left|Z^M_{r_1} \cap Y\right| = 2$. Hence we may conclude that $H' \subseteq U_1$.

Consider now a set $Y \in U_1$, a line $L_0 \in \mathcal{L}_M$ such that $|Y \cap L_0| = 2$ and an element $r \in R$ such that $L_0 = Z^M_r$. Then $Y \cap L_0$ is an edge of $\mathscr{G}(M)$ and so $Y \notin \{\text{3-anticliques of } \mathscr{G}(M)\}$. Moreover, by Proposition 4.1.2, for every line $L \in \mathcal{L}_M \setminus \{L_0\}$, $|L \cap L_0| \leq 1$ and therefore $|L \cap Y| \leq 2$. So $Y \notin \bigcup_{L \in \mathcal{L}_M} P_3(L)$. Therefore $Y \in U_2$ and we may conclude that $U_1 \subseteq U_2$.

Finally, consider a set $Y \in U_2$. There exists a line $J \in L_M$ such that $|Y \cap J| \geq 2$ because Y is not a 3-anticlique of $\mathscr{G}(M)$; and $|Y \cap L| \leq 2$ for every line $L \in \mathcal{L}_M$ because $X \notin \bigcup_{L \in \mathcal{L}_M} P_3(L)$. Then $|Y \cap J| = 2$. Consider now an element $r_0 \in R$ such that $J = Z_{r_0}$ and define y_0 as the single element in the set $Y \setminus (Y \cap J)$. Up to congruence, there exists a subset $R'' \subseteq R$ such that the submatrix $M'' := M[R'', Y \cap J]$ is lower unitriangular, because $Y \cap J \in P_2(V) \subseteq H = H_M$. But $m_{r_0 y_0} = 1$ (because $y_0 \notin J = Z_{r_0}$) and $m_{r_0 y} = 0$ for every $y \in Y \cap J$ (because $J = Z_{r_0}$). Therefore $M[R'' \cup \{r_0\}, Y]$ is congruent to a lower unitriangular matrix and so $Y \in H_M = H$. Hence, we may conclude that $U_2 \subseteq H'$ and therefore the claim holds.

Moreover, $P_{\leq 2}(V) \subseteq H$ because $\mathcal{H} \in \mathrm{BPav}_2$ and so the result holds. □

Proposition 4.1.7. *Given a simplicial complex $\mathcal{H} \in \mathrm{BPav}_2$, there exists a 1-complete matrix in $\mathrm{BR}(\mathcal{H})$.*

Proof. As $P_2(V) \subseteq H$ then, by Lemma 3.3.9, $P_{\leq 1}(V) \subseteq \mathrm{Fl}\,\mathcal{H}$. Then, by definition, $\mathrm{Mat}\,\mathcal{H}$ contains all possible rows with exactly one entry equal to 0, i.e. $\mathrm{Mat}\,\mathcal{H}$ is a 1-complete matrix. Moreover, by Proposition 3.3.6, $\mathrm{Mat}\,\mathcal{H} \in \mathrm{BR}(\mathcal{H})$. □

Proposition 4.1.8. *Given a simplicial complex $\mathcal{H} = (V, H) \in \mathrm{BPav}_2$ and a matrix $M \in \mathrm{BR}(\mathcal{H})$,*

$$\mathrm{Fl}\,\mathcal{H} = P_{\leq 1}(V) \cup \{V\} \cup \mathcal{L}_M \cup \{\textit{superanticliques of } \mathscr{G}(M)\}$$

Proof. Let E denote the set of edges of the graph $\mathscr{G}(M)$. Now, by Lemmas 3.3.8 and 3.3.9, $V \in \mathrm{Fl}\,\mathcal{H}$ and also $P_{\leq 1}(V) \subseteq \mathrm{Fl}\,\mathcal{H}$ (because $P_{\leq 2}(V) \subseteq H$). Moreover, by Lemma 3.3.5, $\mathcal{L}_M \subseteq \mathrm{Fl}\,\mathcal{H}$. Consider now a superanticlique $X \subseteq V$ of the graph $\mathscr{G}(M)$, a subset $I \in 2^X \cap H$ and a point $p \in V \setminus X$. Observe first that $I \in H = H_M$ so $|I| \leq 3$; moreover, I is an anticlique of $\mathscr{G}(M)$ so, by Proposition 4.1.6, $|I| \neq 3$; hence $|I| \leq 2$. We claim that $I \cup \{p\} \in H$. If $|I| \leq 1$ then $I \cup \{p\} \in P_{\leq 2}(V) \subseteq H$ and the claim holds. If $|I| = 2$, then $\mathcal{N}_{\mathscr{G}(M)}(I) = V \setminus X$ because $I \in P_2(X)$ and X is a superanticlique of the graph $\mathscr{G}(M)$. Therefore $xp \in E$ for some element $x \in I$. But then there exists a line $L \in \mathcal{L}_M$ such that $L \cap (I \cup \{p\}) = 2$ and so, by Proposition 4.1.6,

$I \cup \{p\} \in H_M = H$ and the claim also holds in this case. Hence we may conclude that

$$P_{\leq 1}(V) \cup \{V\} \cup \mathcal{L}_M \cup \{\text{superanticliques of } \mathscr{G}(M)\} \subseteq \operatorname{Fl}\mathcal{H}$$

Consider now a flat $F \in \operatorname{Fl}\mathcal{H} \setminus (P_{\leq 1}(V) \cup \{V\} \cup \mathcal{L}_M)$ and distinct elements $x, y \in F$. We claim first that $xy \notin E$. To prove the claim, suppose that $xy \in E$. Then there exists a line $L \in \mathcal{L}_M$ such that $xy \subseteq L$ and so $|L \cap F| \geq 2$. This contradicts Proposition 4.1.2 because $L, F \in \operatorname{Fl}\mathcal{H}$ and so the first claim holds. We may then conclude that F is an anticlique of the graph $\mathscr{G}(M)$ and therefore $\mathcal{N}_{\mathscr{G}(M)}(x) \cup \mathcal{N}_{\mathscr{G}(M)}(y) \subseteq V \setminus F$. We claim that the equality $\mathcal{N}_{\mathscr{G}(M)}(x) \cup \mathcal{N}_{\mathscr{G}(M)}(y) = V \setminus F$ holds. To prove the claim, consider an element $z \in V \setminus F$ and suppose that xyz is an anticlique. Then, by Proposition 4.1.6, $xyz \notin H$. This is a contradiction, because $xy \in P_2(V) \subseteq H$, $xy \subseteq F$, $z \in V \setminus F$ and F is a flat. So the second claim holds. Therefore F is a superanticlique of the graph $\mathscr{G}(M)$ and we may conclude that

$$\operatorname{Fl}\mathcal{H} \subseteq P_{\leq 1}(V) \cup \{V\} \cup \mathcal{L}_M \cup \{\text{superanticliques of } \mathscr{G}(M)\}$$

so the desired equality holds. □

Proposition 4.1.9. *Given a simplicial complex $\mathcal{H} \in \operatorname{Pav}_2$, $\mathcal{H}$ is a matroid if and only if $\mathscr{G}(\operatorname{Mat}\mathcal{H})$ is complete.*

Proof. Write $\mathcal{H} = (V, H)$ and let E denote the set of edges of the graph $\mathscr{G}(\operatorname{Mat}\mathcal{H})$. Suppose that $\mathcal{H}$ is a matroid, consider distinct points $x, y \in V$ and suppose that $xy \notin E$. Then $xy \nsubseteq L$ for every line $L \in \mathcal{L}_{\operatorname{Mat}\mathcal{H}}$. i.e. $xy \nsubseteq F$ for every $F \in \operatorname{Fl}\mathcal{H} \setminus (P_{\leq 1}(V) \cup \{V\})$. But then $\overline{xy} = V$ and so, by Proposition 3.3.22, $xyv \notin H$ for every element $v \in V \setminus \{xy\}$. Hence $xy \in H$ is a facet. This is a contradiction because $\mathcal{H}$ is pure, by Lemma 3.4.7. So $xy \in E$ and we may conclude that $\mathscr{G}(\operatorname{Mat}\mathcal{H})$ is a complete graph.

Conversely, suppose that $\mathscr{G}(\operatorname{Mat}\mathcal{H})$ is a complete graph, consider simplexes $X, Y \in H$ such that $|Y| = |X| + 1$ and assume that $X \cup \{y\} \notin H$ for every element $y \in Y \setminus X$. Then $|X| = 2$ because $\mathcal{H} \in \operatorname{Pav}_2$. So $X \in E$ and therefore there exists a flat $F \in \operatorname{Fl}\mathcal{H} \setminus (P_{\leq 1}(V) \cup \{V\})$ such that $X \subseteq F$. But then $Y \subseteq \overline{X}$ by Proposition 3.3.20, and therefore $Y \subseteq F$, a contradiction because Y is a facet and so, by Proposition 3.3.18, $\overline{Y} = V$. □

4.2 Paving Matroids

Definition 4.2.1. Given $n \in \mathbb{N}_0$ and a simplicial complex $\mathcal{H}$, let $\operatorname{Fl}_n\mathcal{H} := \bigcup_{F \in \operatorname{Fl}\mathcal{H} \setminus \{V\}} P_n(F)$.

Proposition 4.2.2. *Every simplicial complex $\mathcal{H} = (V, H) \in \operatorname{Pav}_n$ such that $P_{n+1}(V) \subseteq \operatorname{Fl}_{n+1}(\mathcal{H}) \cup H$ is boolean representable.*

Proof. Consider a simplicial complex $\mathcal{H} = (V, H) \in \operatorname{Pav}_n$ satisfying $P_{n+1}(V) \subseteq \operatorname{Fl}_{n+1}\mathcal{H} \cup H$. Consider also a simplex $X \in H \setminus \{\varnothing\}$ and fix an element $x \in X$. We claim that $x \notin \overline{X \setminus \{x\}}$. Suppose first that $|X| \leq n$. Then $X \setminus \{x\} \in P_{\leq n-1}(V)$. So, by Lemma 3.3.9, $X \setminus \{x\} \in \operatorname{Fl}\mathcal{H}$. Therefore $\overline{X \setminus \{x\}} = X \setminus \{x\}$ and the claim holds clearly. We may then assume that $|X| = n + 1$.

Suppose now that $(X \setminus \{x\}) \cup \{v\} \in H$ for every $v \in V \setminus X$. Consider a subset $Y \in 2^{X\setminus\{x\}} \cap H$ and an element $p \in V \setminus (X \setminus \{x\})$. If $p = x$ then $Y \cup \{p\} \subseteq X \in H$ and therefore $Y \cup \{p\} \in H$. If $p \in V \setminus X$, then $Y \cup \{p\} \subseteq (X \setminus \{x\}) \cup \{p\} \in H$ and therefore $Y \cup \{p\} \in H$. We may then conclude that $X \setminus \{x\} \in \mathrm{Fl}\,\mathcal{H}$ and so the claim also holds in this case. We may now assume that there exists an element $v_0 \in V \setminus X$ such that $(X \setminus \{x\}) \cup \{v_0\} \in P_{n+1}(V) \setminus H$. Define $Y := (X \setminus \{x\}) \cup \{v_0\}$. So, $Y \in \mathrm{Fl}_{n+1}\mathcal{H}$ and therefore there exists $F \in \mathrm{Fl}\,\mathcal{H} \setminus \{V\}$ such that $Y \in P_{n+1}(F)$. If $x \in F$, then $X \in 2^F \cap H$ and therefore, $X \cup \{q\} \in H$ for every $q \in V \setminus F$ (because F is a flat), a contradiction because $|X| = n+1$ and $\dim \mathcal{H} = n$. Hence $x \notin F$. But $X \setminus \{x\} \subseteq Y \subseteq F$, so $\overline{X \setminus \{x\}} \subseteq F$. Hence $x \notin \overline{X \setminus \{x\}}$ and the claim holds also in this case. So, by Proposition 3.4.6, we may conclude that $\mathcal{H}$ is boolean representable. $\square$

Remark 4.2.3. Given a matroid $\mathcal{H} = (V, H)$, a simplex $X \in H$ and an element $v \in V \setminus X$, the following condition holds: $X \cup \{v\} \in H$ if and only if $v \notin \overline{X}$.

Proof. Consider a simplex $X \in H$ and an element $v \in V \setminus X$. The following statements are equivalent:

1. $X \cup \{v\} \in H$
2. $I \cup \{v\} \in H$ for every $I \in 2^X$
3. $I \cup \{v\} \in H$ for every $I \in H \cap 2^X$
4. $v \notin \overline{X}$

Statement 3 implies statement 2 because $X \in H$. The remaining sequential equivalences are clear. $\square$

Proposition 4.2.4. *Given $n \in \mathbb{N}$ and a simplicial complex $\mathcal{H} = (V, H) \in \mathrm{Pav}_n$, the following conditions are equivalent:*

1. $\mathcal{H}$ is a matroid

2. $P_{n+1}(V) \subseteq \mathrm{Fl}_{n+1}\mathcal{H} \cup H$

Proof. Suppose that condition 1 holds and consider a set $X \in P_{n+1}(V) \setminus H$. Fix an element $x \in X$ and define $X' := X \setminus \{x\}$. By Proposition 3.4.8, $\mathcal{H}$ is boolean representable, so we may use Proposition 3.4.6 and conclude that $x \in \overline{X'}$ (because $X \notin H$). We claim that $\overline{X'} \neq V$. To prove the claim, observe that $X' \in P_n(V)$ and so $X' \in H$ because $\mathcal{H}$ is paving. But, by Lemma 3.4.7, $|Y| = n+1$ for every $Y \in \mathrm{fct}\mathcal{H}$ and therefore $X' \notin \mathrm{fct}\mathcal{H}$. So, there exists an element $v \in V \setminus X$ such that $X' \cup \{v\} \in H$. Hence, by Remark 4.2.3, $v \notin \overline{X'}$ and so the claim holds. Therefore, $\overline{X'} \in \mathrm{Fl}\,\mathcal{H} \setminus \{V\}$ and $X \in P_{n+1}(\overline{X'})$. So $X \in \mathrm{Fl}_{n+1}\mathcal{H}$ and we may conclude that condition 2 holds.

Conversely, suppose that condition 2 holds and consider simplexes $X, Y \in H$ such that $|Y| = |X| + 1$. We claim that there exists an element $y_0 \in Y \setminus X$ such that $X \cup \{y_0\} \in H$. Observe that, if $X \subseteq Y$ then the claim holds trivially. Moreover, if $|X| \leq n-1$ then, for every $y \in Y \setminus X$, $X \cup \{y\} \in P_{\leq n}(V)$ and therefore $X \in H$ because $\mathcal{H}$ is paving, so the claim also holds trivially in this case. We may then assume that $|X| = n$ and $|Y| = n+1$ and $X \nsubseteq Y$. Define $Z := Y \setminus X$ and

$k := |Z|$. Observe that $k \geq 2$ (because $X \nsubseteq Y$) and write $Z = z_1 \dots z_k$. Define sets $X_i := X \cup \{z_i\}$ for every $i \in [k]$. Suppose that $X_i \in \mathrm{Fl}_{n+1}\mathcal{H}$ for every $i \in [k]$. Consider flats $F_i \in \mathrm{Fl}\,\mathcal{H} \setminus \{V\}$ such that $X_i \in P_{n+1}(F_i)$ for every $i \in [k]$. For all distinct indices $i, j \in [k]$, $X_i \cap X_j \subseteq F_i \cap F_j$ and $X_i \cap X_j = X$; therefore $|X_i \cap X_j| = n$ and $|F_i \cap F_j| \geq n$; moreover, by Proposition 4.1.2 we may conclude that $F_i = F_j$. Hence, there exists a flat $F \in \mathrm{Fl}\,\mathcal{H} \setminus \{V\}$ such that $X \cup Z \subseteq F$ and therefore $Y \subseteq F$. But then, for every $v \in V \setminus F$, $Y \cup \{v\} \in H$ (because F is a flat and $Y \in 2^F \cap H$), a contradiction because $|Y| = n+1$ and $\dim \mathcal{H} = n$. So we may conclude that $X_i \in H$ for some $i \in [k]$ and therefore the claim holds by taking $y_0 = z_i$. Then $\mathcal{H}$ satisfies the exchange property and condition 1 holds. □

Proposition 4.2.5. *Given $n \in \mathbb{N}$ and a simplicial complex $\mathcal{H} = (V, H) \in \mathrm{Pav}_n$, the following conditions are equivalent:*

1. $\mathcal{H}$ is a matroid such that $|F| \leq n+1$ for every $F \in \mathrm{Fl}\,\mathcal{H} \setminus \{V\}$

2. $P_{n+1}(V) \subseteq \mathrm{Fl}\,\mathcal{H} \cup H$

Proof. Suppose that condition 1 holds. Then, by Proposition 4.2.4, $P_{n+1}(V) \subseteq \mathrm{Fl}_{n+1}\mathcal{H} \cup H$. But in this case $\mathrm{Fl}_{n+1}\mathcal{H} \subseteq \mathrm{Fl}\,\mathcal{H}$ because $|F| \leq n+1$ for every $F \in \mathrm{Fl}\,\mathcal{H} \setminus \{V\}$. Therefore condition 2 holds.

Now, observe that $\mathrm{Fl}\,\mathcal{H} \cap P_{n+1}(V) \subseteq \mathrm{Fl}_{n+1}(\mathcal{H})$, so if $\mathcal{H}$ satisfies condition 2 then $\mathcal{H}$ also satisfies $P_{n+1}(V) \subseteq \mathrm{Fl}_{n+1}\mathcal{H} \cup H$. Moreover, by the previous observation and Proposition 4.2.4, $\mathcal{H}$ is a matroid. Suppose that there exists a flat $F \in \mathrm{Fl}\,\mathcal{H} \setminus \{V\}$ such that $|F| \geq n+2$. Consider $X \in P_{n+1}(F)$. Observe that $X \subset F$ and therefore $X \cap F = X$ and so $|X \cap F| = n+1$. Hence, by Lemma 4.1.2, $X \notin \mathrm{Fl}\,\mathcal{H}$. Then, by condition 2, $X \in H$. Therefore $X \cup \{v\}$ for every $v \in V \setminus F$ (because $F \in \mathrm{Fl}\,\mathcal{H}$ and $X \in 2^F \cap H$). This is a contradiction, because $|X| = n+1$ and $\dim \mathcal{H} = n$. □

4.2.1 Dimension 2: Examples

Proposition 4.2.6. *Given a finite field K and elements $a, b, c \in K$ not all zero, define the simplicial complex $\mathcal{H}_{a,b,c} = (V, H_{a,b,c}) \in \mathrm{Pav}_2$ such that $V = K$ and, for every subset $X \in P_3(V)$, $X \in H_{a,b,c}$ if and only if there exist an enumeration $X = xyz$ such that $ax + by + cz \neq 0$. Then $\mathcal{H}_{a,b,c}$ is a matroid.*

Proof. Write $H := H_{a,b,c}$ and $\mathcal{H} := \mathcal{H}_{a,b,c}$. Consider a set $X \in P_3(V) \setminus H$. We claim that $X \in \mathrm{Fl}\,\mathcal{H}$. If $X = V$ then the claim holds trivially. Consider then a subset $I \in 2^X \cap H$ and an element $v \in V \setminus X$. Observe first that, if $|I| \leq 1$ then $I \cup \{v\} \in P_{\leq 2}(V)$ and therefore $I \cup \{v\} \in H$ (because $\mathcal{H}$ is paving). Then we may assume that $|I| = 2$ (because $I \in 2^X \cap H$ and $X \notin H$). Write $I = i_1 i_2$, let x denote the single element in the set $X \setminus I$ and assume without loss of generality that $a \neq 0$. Then $ax + bi_1 + ci_2 = 0$ (because $xi_1i_2 = X \notin H$) and so the following equalities hold: $av + bi_1 + ci_2 = (ax + bi_1 + ci_2) + a(v - x) = a(v - x)$. But then $av + bi_1 + ci_2 \neq 0$ (because $a \neq 0$ and $v \neq x$). So $I \cup \{v\} \in H$ and therefore the claim holds. Hence we may conclude that $P_3(V) \subseteq \mathrm{Fl}\,\mathcal{H} \cup H$ and therefore $\mathcal{H}$ is a matroid by Proposition 4.2.5. □

Definition 4.2.7. Let K be a finite field. For all $x_0, x_1, x_2, y_0, y_1, y_2 \in K$, write $(x_0, x_1, x_2) \sim (y_0, y_1, y_2)$ if there exists an element $\lambda \in K \setminus \{0\}$ such that $(x_0, x_1, x_2) = \lambda(y_0, y_1, y_2)$. Then $\sim$ defines an equivalence relation in $K^3 \setminus \{(0,0,0)\}$. Define the *projective space of dimension* 2 *over* K as the set of equivalence classes $\mathbb{P}^2(K) := (K^3 \setminus \{(0,0,0)\})/\sim$. For all $a, b, c \in K$ not all zero, use $(a:b:c)$ to denote the equivalence class of the element $(a, b, c) \in K^3 \setminus \{(0,0,0)\}$ and define the *projective line* associated to $(a:b:c)$ as $L_{(a:b:c)} := \{(x:y:z) \in \mathbb{P}^2(K) \mid ax + by + cz = 0\}$.

Proposition 4.2.8. *Given a finite field K define the boolean matrix $M = (m_{LX})$ with columns indexed by $\mathbb{P}^2(K)$, rows indexed by the set $\{L_{(a:b:c)} \mid (a:b:c) \in \mathbb{P}^2(K)\}$ and such that $m_{LX} = 0$ if and only if $X \in L$. Define the simplicial complex $\mathcal{H} = (\mathbb{P}^2(K), H_M)$. Then $\mathcal{H} \in \mathrm{Pav}_2$ and $\mathcal{H}$ is a matroid.*

Proof. The following properties hold:

1. Given $(a:b:c) \in \mathbb{P}^2(K)$, $|L_{(a:b:c)}| \geq 2$
2. Given $(a:b:c) \in \mathbb{P}^2(K)$, $L_{(a:b:c)} \neq \mathbb{P}^2(K)$
3. Given distinct elements $X := (x_0 : x_1 : x_2), Y := (y_0 : y_1 : y_2) \in \mathbb{P}^2(K)$, there exists a single element $(a_0 : b_0 : c_0) \in \mathbb{P}^2(K)$ such that $X, Y \in L_{(a_0:b_0:c_0)}$
4. Given distinct elements $(a_0 : b_0 : c_0), (a_1 : b_1 : c_1) \in \mathbb{P}^2(K)$, $|L_{(a_0:b_0:c_0)} \cap L_{(a_1:b_1:c_1)}| \leq 1$

To prove property 1, observe that one of the elements $a, b, c \in K$ is different from 0. Therefore, the set $S := \{(x, y, z) \in K^3 \mid ax + by + cz = 0\}$ is a linear subspace of K^3 of dimension 2. So we may take linearly independent vectors $U = (u_0, u_1, u_2), V = (v_0, v_1, v_2) \in S$. Then $(u_0 : u_1 : u_2) \neq (v_0 : v_1 : v_2)$ and $(u_0 : u_1 : u_2), (v_0 : v_1 : v_2) \in L_{(a:b:c)}$.

To prove property 2, observe that, for every $(a:b:c) \in \mathbb{P}^2(K)$, one of the elements $(1:0:0), (0:1:0)$ and $(0:0:1)$ does not lie on the line $L_{(a:b:c)}$.

To prove property 3, consider the following system of equations in variables a, b, c taking values in K:

$$\begin{cases} ax_0 + bx_1 + cx_2 = 0 \\ ay_0 + by_1 + cy_2 = 0 \end{cases}$$

The coefficient matrix associated to this system of equations is:

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{pmatrix}$$

Observe that M has maximal rank because X and Y are distinct elements of $\mathbb{P}^2(K)$. Let S denote the solution set of the system above. So S is a linear subspace of K^3 of dimension 1. Hence $|(S \setminus \{(0,0,0)\})/\sim| = 1$ and property 3 holds.

To prove property 4, suppose that there exist distinct elements $X, Y \in L_{(a_0:b_0:c_0)} \cap L_{(a_1:b_1:c_1)}$. Then, by property 3, $(a_0 : b_0 : c_0) = (a_1 : b_1 : c_1)$.

Now, write $\mathcal{H} = (V, H)$ where $V = \mathbb{P}^2(K)$ and $H = H_M$; and $\mathcal{L} := \{L_{(a:b:c)} \mid (a:b:c) \in V\}$. For every $(a:b:c) \in V$, $2 \leq |L_{(a:b:c)}| < |V|$ by properties 1 and 2, so $\mathcal{L} \subseteq \mathcal{L}_M$. Then

$\mathcal{L} \subseteq \mathrm{Fl}\,\mathcal{H} \setminus \{V\}$ by Lemma 3.3.5. Given distinct elements $X, Y, Z \in \mathbb{P}^2(V)$, there exists a line $L_{XY} \in \mathcal{L}$ such that $XY \subseteq L$ (property 3). If $Z \in L_{XY}$, then $XYZ \in \mathrm{Fl}_3\mathcal{H}$. If $Z \notin L_{XY}$ then consider lines $L_{XZ}, L_{YZ} \in \mathcal{L}$ such that $XZ \subseteq L_{XZ}$ and $YZ \subseteq L_{YZ}$ (property 3). Moreover, $X \notin L_{YZ}$, $Y \notin L_{XZ}$ and $Z \notin L_{XY}$ (property 4). Therefore, the submatrix of M with rows and columns sequentially indexed by L_{YZ}, L_{XZ}, L_{XY} and X, Y, Z, respectively, is lower unitriangular and therefore $XYZ \in H$. Using the previous properties and this argument, we may conclude that: $P_2(V) \subseteq H$; there exists an element $X \in P_3(V) \cap H$; $P_4(V) \cap H = \varnothing$; and $P_3(V) \subseteq \mathrm{Fl}_3\mathcal{H} \cup H$. Therefore, by Proposition 4.2.5, $\mathcal{H}$ is a matroid. □

4.3 A Metric in $\mathrm{BPav}_2/\cong$

Context 4.3.1. Throughout this Section, we aim at introducing two notions of distance between simplicial complexes lying in $\mathrm{BPav}_2 \cup \{\mathcal{U}_{2,n} \mid n \in \mathbb{N}\}$ (given $n \in \mathbb{N}$, $\mathcal{U}_{2,n}$ refers to the uniform matroid introduced in Definition 3.5.3).

Note 4.3.2. Throughout this Section and Chapter 6, we assume that $\mathcal{U}_{2,n} \in \mathrm{BPav}_2^{(n)}$ for every $n \in \mathbb{N}$.

Notation 4.3.3. Given $n \in \mathbb{N}$, use $\mathrm{BPav}_2/\cong$ (or $\mathrm{BPav}_2^{(\mathrm{n})}/\cong$) to denote the set of all isomorphism classes of boolean representable paving simplicial complexes of dimension 2 (with n vertices). To simplify notation, we use the simplicial complex $\mathcal{H} \in \mathrm{BPav}_2$ to denote its isomorphism class.

Definition 4.3.4. Given arbitrary sets A and B, the *symmetric difference* of A and B is $A \triangle B := (A \cup B) \setminus (A \cap B)$.

Definition 4.3.5. Given $n \in \mathbb{N}$, define the set $D_n \subseteq P_2\left(\mathrm{BPav}_2^{(n)}/\cong\right)$ determined by the following condition: $\mathcal{H}\mathcal{H}' \in D_n$ if and only if there exist matrices $M \in \mathrm{BR}_{\text{1-complete}}(\mathcal{H})$ and $M' \in \mathrm{BR}_{\text{1-complete}}(\mathcal{H}')$ such that $|\mathcal{L}_M \triangle \mathcal{L}_{M'}| = 1$. Define now the graph $\mathcal{D}_n = \left(\mathrm{BPav}_2^{(n)}/\cong, D_n\right)$.

Proposition 4.3.6. *Given $n \in \mathbb{N}$, the graph $\mathcal{D}_n$ is connected.*

Proof. Consider simplicial complexes $\mathcal{H} \neq \mathcal{H}' \in \mathrm{BPav}_2^{(n)}/\cong$ and matrices $M \in \mathrm{BR}_{\text{1-complete}}(\mathcal{H})$, $M' \in \mathrm{BR}_{\text{1-complete}}(\mathcal{H}')$ (there exist such matrices by Proposition 4.1.7). Define $n := |\mathcal{L}_M|$, $m := |\mathcal{L}_{M'}|$ and assume that the first n rows of M (respectively the first m rows of M') correspond to the elements of $\mathcal{L}_M$ (respectively elements of $\mathcal{L}_{M'}$). Now, define matrices:

- $M_0 := M$
- M_i obtained from M_{i-1} by removing the i^{th} row of the matrix M, for every $i \in [n]$
- M_{n+j} obtained from M_{n+j-1} by adding the j^{th} row of M', for every $j \in [m]$

Note that the matrix M_k is uniquely determined up to congruence, for every $k \in [n+m]$. Also, we get $M_{n+m} = M'$. Define the set

$$A := \left\{k \in [n+m] \mid \mathcal{H}_{M_{k-1}} \neq \mathcal{H}_{M_k}\right\}$$

Now, $A \neq \varnothing$ because $\mathcal{H} \neq \mathcal{H}'$. Moreover, for every $k \in A$, $\mathcal{H}_{M_{k-1}}\mathcal{H}_{M_k} \in D_n$ because $\left|\mathcal{L}_{M_{k-1}} \triangle \mathcal{L}_{M_k}\right| = 1$. Define $l := |A|$ and consider an enumeration $A = k_1 \dots k_l$ with the usual order. Therefore $\mathcal{H} = \mathcal{H}_M = \mathcal{H}_{M_0}$, $\mathcal{H}' = \mathcal{H}_{M'} = \mathcal{H}_{M_l}$ and

$$\left(\mathcal{H}, \mathcal{H}_{M_{k_1}}, \mathcal{H}_{M_{k_2}}, \dots, \mathcal{H}_{M_{k_{l-1}}}, \mathcal{H}'\right)$$

is a path in $\mathcal{D}_n$ between vertices $\mathcal{H}$ and $\mathcal{H}'$. So we may conclude that the set of paths $\mathcal{P}_{\mathcal{H},\mathcal{H}'}(\mathcal{D}_n) \neq \varnothing$ for all distinct $\mathcal{H}, \mathcal{H}' \in \mathrm{BPav}_2^{(n)}/\cong$ and therefore the graph $\mathcal{D}_n$ is connected. □

Definition 4.3.7. Given $n \in \mathbb{N}$, define the map

$$\begin{array}{rrcl} d_n : & \left(\mathrm{BPav}_2^{(n)}/\cong\right) \times \left(\mathrm{BPav}_2^{(n)}/\cong\right) & \to & \mathbb{N}_0 \\ & (\mathcal{H}, \mathcal{H}') & \mapsto & \min_{P \in \mathcal{P}_{\mathcal{H},\mathcal{H}'}(\mathcal{D}_n)} \mathrm{len}(P) \end{array}$$

Remark 4.3.8. Given $n \in \mathbb{N}$, the map d_n is well defined because $\mathcal{H} \in \mathcal{P}_{\mathcal{H},\mathcal{H}}(\mathcal{D}_n)$ for every $\mathcal{H} \in \mathrm{BPav}_2^{(n)}/\cong$ and $\mathcal{P}_{\mathcal{H},\mathcal{H}'}(\mathcal{D}_n) \neq \varnothing$ for all distinct $\mathcal{H}, \mathcal{H}' \in \mathrm{BPav}_n(2)$, by Proposition 4.3.6.

Proposition 4.3.9. *The map d_n is a metric in $\mathrm{BPav}_2^{(n)}/\cong$.* □

Definition 4.3.10. Define the set $D \subseteq P_2(\mathrm{BPav}_2/\cong)$ determined by the following condition: $\mathcal{H}\mathcal{H}' \in D$ if and only if one of the following properties hold:

- $\mathcal{H}, \mathcal{H}' \in \mathrm{BPav}_2^{(n)}/\cong$ for some $n \in \mathbb{N}$ and $\mathcal{H}\mathcal{H}' \in D_n$
- $\mathcal{H} \in \mathrm{BPav}_2^{(n)}/\cong$, $\mathcal{H} \in \mathrm{BPav}_2^{(n+1)}$ for some $n \in \mathbb{N}$ and there exist matrices $M \in \mathrm{BR}_{\text{1-complete}}(\mathcal{H})$, $M' \in \mathrm{BR}_{\text{1-complete}}(\mathcal{H}')$ such that

$$M' \cong \left(\begin{array}{ccc|c} & & & 1 \\ & M & & \vdots \\ & & & 1 \\ \hline 1 & \dots & 1 & 0 \end{array}\right)$$

Define now the graph $\mathcal{D} = (\mathrm{BPav}_2/\cong, D)$.

Remark 4.3.11. $D_n \subseteq D$ for every $n \in \mathbb{N}$. □

Proposition 4.3.12. *The graph $\mathcal{D}$ is connected.*

Proof. Consider simplicial complexes $\mathcal{H} \neq \mathcal{H}' \in \mathrm{BPav}_2/\cong$ such that $\mathcal{H} \in \mathrm{BPav}_2^{(n)}/\cong$ and $\mathcal{H}' \in \mathrm{BPav}_2^{(m)}/\cong$ for some $n, m \in \mathbb{N}$ with $n \geq m$. Consider also matrices $M \in \mathrm{BR}_{\text{1-complete}}(\mathcal{H})$ and $M' \in \mathrm{BR}_{\text{1-complete}}(\mathcal{H}')$ (there exist such matrices by Proposition 4.1.7). Define matrices N and N' obtained from M and M' by removing the rows that correspond to elements of $\mathcal{L}_M$ and $\mathcal{L}_{M'}$, respectively. Then, by Remark 4.1.4, $\mathcal{H}_N = \mathcal{U}_{2,n} \in \mathrm{BPav}_2^{(n)}/\cong$ and $\mathcal{H}_{N'} = \mathcal{U}_{2,m} \in \mathrm{BPav}_2^{(m)}/\cong$. Therefore, by Proposition 4.3.6 and Remark 4.3.11, there exist paths $P \in \mathcal{P}_{\mathcal{H},\mathcal{U}_{2,n}}(\mathcal{D}_n) \subseteq \mathcal{P}_{\mathcal{H},\mathcal{U}_{2,n}}(\mathcal{D})$ and $Q \in \mathcal{P}_{\mathcal{H}',\mathcal{U}_{2,m}}(\mathcal{D}_m) \subseteq \mathcal{P}_{\mathcal{H}',\mathcal{U}_{2,m}}(\mathcal{D})$.

Now, observe that

$$N = \left(\begin{array}{cccc|ccc} 0 & 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 & 1 & \dots & 1 \\ \hline 1 & 1 & \dots & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 1 & \dots & 0 \end{array}\right) = \left(\begin{array}{cccc|ccc} & & & & 1 & \dots & 1 \\ & & N' & & 1 & \dots & 1 \\ & & & & \vdots & \ddots & \vdots \\ & & & & 1 & \dots & 1 \\ \hline 1 & 1 & \dots & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 1 & \dots & 0 \end{array}\right)$$

So, by a clear inductive argument using the second property of Definition 4.3.10, there exists a path in $\mathcal{D}$ between $\mathcal{U}_{2,n}$ and $\mathcal{U}_{2,m}$. Therefore, there exists a path in $\mathcal{D}$ between $\mathcal{H}$ and $\mathcal{H}'$. Hence we may conclude that the set of paths $\mathcal{P}_{\mathcal{H},\mathcal{H}'}(\mathcal{D}) \neq \varnothing$ for all distinct $\mathcal{H}, \mathcal{H}' \in \mathrm{BPav}_2/\cong$ and therefore the graph $\mathcal{D}$ is connected. $\square$

Definition 4.3.13. Define the map

$$\begin{array}{rrcl} d: & (\mathrm{BPav}_2/\cong)\times(\mathrm{BPav}_2/\cong) & \to & \mathbb{N}_0 \\ & (\mathcal{H},\mathcal{H}') & \mapsto & \min_{P\in\mathcal{P}_{\mathcal{H},\mathcal{H}'}(\mathcal{D})} \mathrm{len}(P) \end{array}$$

Remark 4.3.14. The map d is well defined because $\mathcal{H} \in \mathcal{P}_{\mathcal{H},\mathcal{H}}(\mathcal{D})$ for every $\mathcal{H} \in \mathrm{BPav}_2/\cong$ and $\mathcal{P}_{\mathcal{H},\mathcal{H}'}(\mathcal{D}) \neq \varnothing$ for all distinct $\mathcal{H}, \mathcal{H}' \in \mathrm{BPav}_2/\cong$ by Proposition 4.3.12.

Proposition 4.3.15. *The map d is a metric in $\mathrm{BPav}_2/\cong$.* $\square$

Chapter 5

Disjoint Representations of Simplicial Complexes

5.1 Short Representations

Context 5.1.1. Throughout this Section, we aim at getting a full description of simplicial complexes in BPav_2 admitting 1-complete boolean representations with disjoint sets of lines, being one of them short. So, up until Proposition 5.1.12, let $\mathcal{H} = (V, H) \in \mathrm{BPav}_2$, consider $M, N \in \mathrm{BR}_{\text{1-complete}}(\mathcal{H})$ such that $M \in \mathrm{BR}_{\text{short}}(\mathcal{H})$ and $\mathcal{L}_M \cap \mathcal{L}_N = \varnothing$. Let Γ denote the graph $\mathscr{G}(M)$ and E denote its edge set.

Remark 5.1.2. The following equalities hold:

$$\begin{aligned} H_M &= P_{\leq 2}(V) \cup \{X \in P_3(V) \mid |X \cap L| = 2 \text{ for some } L \in \mathcal{L}_M\} \\ &= P_{\leq 3}(V) \setminus \{\text{3-anticliques of } \mathscr{G}(M)\} \end{aligned}$$

Moreover, $|V| \geq 3$ and $E \neq \varnothing$.

Proof. The first two equalities follow directly from Proposition 4.1.6 by observing that $\bigcup_{L \in \mathcal{L}_M} P_3(L) = \varnothing$ because M is a short matrix. The final assertion holds because $\dim \mathcal{H} = 2$ and so, by the first equality, $\mathcal{L}_M \neq \varnothing$. Then $E = \bigcup_{L \in \mathcal{L}_M} P_2(L) \neq \varnothing$. □

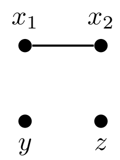
Lemma 5.1.3. *Given a subset $X \in P_4(V)$, one of the following conditions holds:*

1. *X is an anticlique of Γ*

2. *The number of edges of $\Gamma[X]$ is at least 2*

In particular, there exists at most one vertex of degree 0 in Γ.

Proof. Suppose that none of the conditions hold and write $X = x_1x_2yz$ such that $x_1x_2 \in E$ is the only edge in the subgraph $\Gamma[X]$. We get the following configuration:



We use now both characterizations of $H = H_M = H_N$ given by Proposition 4.1.6 and the properties of the closure operator given by Definition 3.3.17. Note that x_1x_2y is not a 3-anticlique of Γ and so $x_1x_2y \in H_M = H = H_N$. Then there exists a line $J \in \mathcal{L}_N$ such that $|x_1x_2y \cap J| = 2$. But both $x_1x_2 \in \mathcal{L}_M$ and $J \in \mathcal{L}_N$ are flats of $\mathcal{H}$ and therefore $|x_1x_2 \cap J| \leq 1$ by Proposition 4.1.2. Therefore, we may fix the element $x \in x_1x_2$ such that $xy \subseteq J$. Now, for every index $i \in [2]$, x_iyz is a 3-anticlique of Γ and so $x_iyz \notin H$. Then, by Proposition 3.3.20, $x_1, x_2 \in \overline{yz}$ and $z \in \overline{xy}$. But then $\overline{yz} = V$, because $x_1x_2y \in 2^{\overline{yz}} \cap \mathrm{fct}\mathcal{H}$; and $\overline{xy} = V$, because $yz \subseteq \overline{xy}$. This contradicts the minimality of $\overline{xy}$ because $xy \subseteq J \subset V$.

Suppose now that there exist two distinct vertices $x, y \in V$ of degree 0. By Remark 5.1.2, $E \neq \varnothing$ and therefore there exist distinct vertices $z, t \in V \setminus xy$ such that $zt \in E$. But then the subgraph $\Gamma[xyzt]$ contains exactly one edge: a contradiction. □

Lemma 5.1.4. Γ *is triangle-free.*

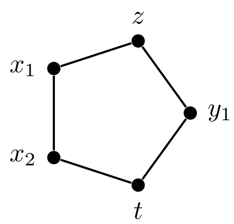
Proof. Suppose that there exists a subset $X \in P_3(V)$ such that $\Gamma[X]$ is a triangle. Then $P_2(X) \subseteq \mathcal{L}_M$. Moreover, X is not a 3-anticlique of Γ and so $X \in H$. Then there exists a line $J \in \mathcal{L}_N$ such that $|X \cap J| = 2$. But $X \cap J \in P_2(X) \subseteq \mathcal{L}_M$ and $J \in \mathcal{L}_N$ are flats of $\mathcal{H}$ intersecting in two points, contradicting Proposition 4.1.2. □

Lemma 5.1.5. *One of the following conditions holds:*

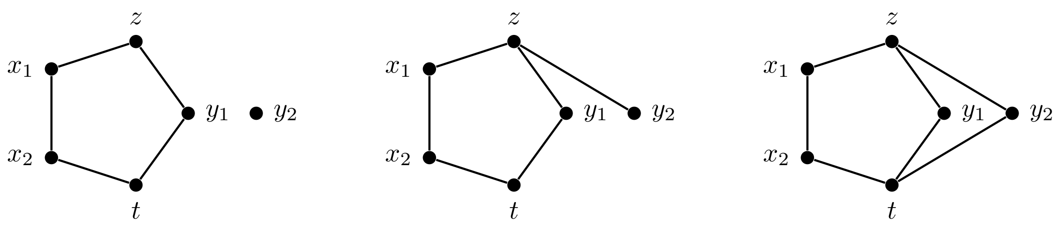
1. Γ *is isomorphic to* $\mathcal{C}_5$

2. *There are no cycles of length* 5 *in* Γ

Proof. Suppose that there exists a cycle of length 5 in Γ with vertex set $X \in P_5(V)$. Write $X = x_1x_2y_1zt$ and assume the edges lying on the cycle are as follows:



Suppose now that condition 1 does not hold and observe that the only edges in the subgraph $\Gamma[X]$ are represented in the picture above, because Γ is triangle-free by Lemma 5.1.4. Then there exists an extra vertex $y_2 \in V \setminus x_1x_2y_1zt$. We get one of the following configurations:



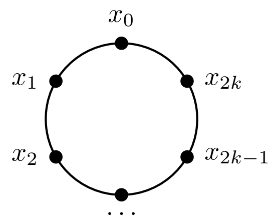
Note that in all configurations $\Gamma[x_1x_2y_1y_2]$ has exactly one edge, contradicting Lemma 5.1.3. $\square$

Lemma 5.1.6. *One of the following conditions holds:*

1. *Γ is isomorphic to $\mathcal{C}_5$*

2. *There are no cycles of odd length in Γ*

Proof. Suppose that condition 1 does not hold. Proceed by induction on k to prove that there are no cycles of length $2k+1$ for every $k \in \mathbb{N}$, the cases $k=1$ and $k=2$ being immediate from Lemmas 5.1.4 and 5.1.5. Suppose now that $k \geq 3$; that there are no cycles of length $2m+1$ for every $m < k$; and that there exists a cycle of length $2k+1$ in Γ with vertex set $X \in P_{2k+1}(V)$. Write $X = x_0x_1 \ldots x_{2k}$ and suppose that the edges lying on the cycle are as follows:



We claim that the subgraph $\Gamma[x_0x_2x_{k+1}x_{k+2}]$ contains exactly one edge. To prove the claim, observe first that $x_0x_2 \notin E$ by Lemma 5.1.4. Moreover,

- By adding the edge x_0x_{k+1} or the edge x_2x_{k+2} to the cycle above, two new cycles appear of lengths $k+1$ and $k+2$
- By adding the edge x_0x_{k+2} or the edge x_2x_{k+1} to the cycle above, two new cycles appear of lengths k and $k+3$

Now, $k < k+1 < k+2 < k+3 < 2k+1$ because $k \geq 3$ and so in both of the above cases there exists a cycle of odd length $2m+1$ for some $m < k$. Therefore, by the induction hypothesis, $x_0x_{k+1}, x_0x_{k+1} \notin E$ and so the claim holds. This contradicts Lemma 5.1.3 and so we may conclude that Γ does not contain a cycle of length $2k+1$. $\square$

Corollary 5.1.7. *One of the following conditions holds:*

1. *Γ is isomorphic to $\mathcal{C}_5$*

2. *Γ is bipartite*

Proof. Follows immediately from Lemmas 5.1.6 and 2.2.39. $\square$

Definition 5.1.8. Given $n, m \in \mathbb{N}$ with $m+n \geq 3$, define the set $\Omega(m,n)$ of all graphs (up to isomorphism) that can be obtained from the complete bipartite graph $K_{m,n}$ by removing a matching.

Proposition 5.1.9. *If there exists some vertex $v \in V$ with $\deg_\Gamma(v) = 0$, then the subgraph $\Gamma[V \setminus \{v\}]$ is a complete bipartite graph.*

Proof. If there exists such a vertex $v \in V$, then Γ is not isomorphic to $\mathcal{C}_5$ and therefore is a bipartite graph by Corollary 5.1.7. Consider partition sets A and B of Γ with $|A| = m$ and $|B| = n$ for some $m, n \in \mathbb{N}$ with $m \geq 1$ and $n \geq 2$ (there exist such integers because $|V| \geq 3$ by Remark 5.1.2). Moreover, we may assume that $v \in B$ because if $A = \{v\}$ then $E = \varnothing$, contradicting Remark 5.1.2. Now, for every vertex $a \in A$, $\deg_\Gamma(a) \neq 0$ (because v is the only isolated vertex by Lemma 5.1.3); consider a vertex $b_0 \in \mathcal{N}_\Gamma(a)$; then, for every $b \in B \setminus \{b_0\}$, the subgraph $\Gamma[ab_0bv]$ contains at least 2 edges and therefore $ab \in E$; we may then conclude that $ab \in E$ for all $a \in A$ and $b \in B$. □

Proposition 5.1.10. *If there are no vertices of degree 0 in Γ, then one of the following conditions holds:*

1. *Γ is isomorphic to $\mathcal{C}_5$*
2. *$\Gamma \in \Omega(m, n)$ for some $m, n \in \mathbb{N}$ with $m + n \geq 3$*

Proof. Suppose that none of the conditions hold. By Corollary 5.1.7, Γ is a bipartite graph. Consider partition sets A and B with sizes $|A| = m$ and $|B| = n$ and observe that $\Gamma \notin \Omega(m, n)$ because condition 2 does not hold. So the subset $E' := \{ab \mid a \in A, b \in B\} \setminus E$ is not a matching. Therefore there exist two elements $X, Y \in E'$ sharing one vertex. We may then assume that there exist distinct vertices $a \in A$ and $b_1, b_2 \in B$ such that $ab_1, ab_2 \notin E$. Observe now that $\deg_\Gamma(a) \neq 0$ and fix some vertex $b \in \mathcal{N}_\Gamma(a) \subseteq B \setminus b_1b_2$. But then the subgraph $\Gamma[abb_1b_2]$ contains exactly one edge, contradicting Lemma 5.1.3. □

Definition 5.1.11. Given $m, n \in \mathbb{N}$ with $m \leq n$ and $m + n \geq 3$, define simplicial complexes $\mathcal{H}^{(1)}_{m,n} = (V_1, H^{(1)}_{m,n})$ and $\mathcal{H}^{(2)}_{m,n} = (V_2, H^{(2)}_{m,n})$ as follows:

- $V_1 := [m + n]$
- $X_1 := [m]$, $Y_1 := [m + n] \setminus [m]$, $Z_1 := P_3(X_1) \cup P_3(Y_1)$, $H^{(1)}_{m,n} = P_{\leq 3}(V_1) \setminus Z_1$
- $V_2 := [m + n + 1]$
- $X_2 := [m] \cup \{m + n + 1\}$, $Y_2 := [m + n + 1] \setminus [m]$, $Z_2 := P_3(X_2) \cup P_3(Y_2)$, $H^{(2)}_{m,n} = P_{\leq 3}(V_2) \setminus Z_2$

Corollary 5.1.12. *Up to isomorphism, $\mathcal{H}$ is one of the following simplicial complexes:*

1. *$\mathcal{U}_{3,5}$*
2. *$\mathcal{H}^{(1)}_{m,n}$ for some $m, n \in \mathbb{N}$*
3. *$\mathcal{H}^{(2)}_{m,n}$ for some $m, n \in \mathbb{N}$*

Proof. Use the notation of Definition 5.1.11. The number of vertices of $\mathcal{H}$ coincides with the number of vertices of the graph Γ. Moreover, $H = H_M = P_{\leq 3}(V) \setminus \{$3-anticliques of $\Gamma\}$ by Remark 5.1.2. By Propositions 5.1.9 and 5.1.10, one of the following conditions holds:

1. Γ is isomorphic to $\mathcal{C}_5$

2. $\Gamma \in \Omega(m,n)$ for some $m,n \in \mathbb{N}$ with $m+n \geq 3$

3. There exists a vertex v such that $\deg_\Gamma(v) = 0$ and $\Gamma[V \setminus \{v\}]$ is isomorphic to $K_{m,n}$ for some $m,n \in \mathbb{N}$

Now, it remains to observe that:

- If condition 1 holds, then there are no 3-anticliques on Γ and so $\mathcal{H} \cong \mathcal{U}_{3,5}$
- If condition 2 holds and Γ admits partition sets A and B with $|A| = m$ and $|B| = n$ then the set of 3-anticliques of Γ is $P_3(A) \cup P_3(B)$ and so $\mathcal{H} \cong \mathcal{H}^{(1)}_{m,n}$
- If condition 3 holds and $\Gamma[V \setminus \{v\}]$ admits partition sets A and B with $|A| = m$ and $|B| = n$ then the set of 3-anticliques of Γ is $P_3(A \cup \{v\}) \cup P_3(B \cup \{v\})$ and so $\mathcal{H} \cong \mathcal{H}^{(2)}_{m,n}$ □

Context 5.1.13. At this point, we established that, if a simplicial complex in BPav_2 admits alternative boolean representations as desired, then it must lie in the set of simplicial complexes described in Corollary 5.1.12. Now, we must show that, up to isomorphism, this set of simplicial complexes is infinite; and all its elements admit alternative boolean representations as desired.

Lemma 5.1.14. *Given $m,n \in \mathbb{N}$ with $m \leq n$ and $m+n \geq 3$,*

$$\min_{U \in P_2([m+n])} \left|\left\{X \in H^{(1)}_{m,n} \mid U \subseteq X\right\}\right| = \min_{U \in P_2([m+n+1])} \left|\left\{X \in H^{(2)}_{m,n} \mid U \subseteq X\right\}\right| = \min\{m,n\} = m$$

Proof. We use the notation of Definition 5.1.11 and write

$$\mathrm{m}_1 := \min_{U \in P_2([m+n])} \left|\left\{X \in H^{(1)}_{m,n} \mid U \subseteq X\right\}\right|$$

Consider a subset $U \in P_2(V_1)$. One of the following conditions holds:

- If $U = uv$ for some $u \in X_1$ and $v \in Y_1$ then there are $m+n-2$ simplexes containg U, namely $\{U \cup \{v\} \mid v \in V_1 \setminus U\}$
- If $U \in P_2(X_1)$ then there are n simplexes containing U, namely $\{U \cup \{y\} \mid y \in Y_1\}$
- If $U \in P_2(Y_1)$ then there are m simplexes containing U, namely $\{U \cup \{x\} \mid x \in X_1\}$

Now, recall that $m \leq n$ and $m+n \geq 3$, so we may conclude that $P_2(Y_1) \neq \varnothing$ and $n \geq 2$. In particular, $m+n-2 \geq m$. So $\mathrm{m}_1 = m$.

Now, write

$$\mathrm{m}_2 := \min_{U \in P_2([m+n+1])} \left|\left\{X \in H^{(2)}_{m,n} \mid U \subseteq X\right\}\right|$$

Consider a subset $U \in P_2(V_2)$. One of the following conditions holds:

- If $U = uv$ for some $u \in [m]$ and $v \in [m+n] \setminus [m]$ then there are $m+n-1$ simplexes containing U, namely $\{U \cup \{v\} \mid V_2 \setminus U\}$
- If $U \in P_2(X_2)$ then there are n simplexes containing U, namely $\{U \cup \{y\} \mid y \in Y_2 \setminus \{n+m+1\}\}$
- If $U \in P_2(Y_2)$ then there are m simplexes containing U, namely $\{U \cup \{x\} \mid x \in X_2 \setminus \{n+m+1\}\}$

Similarly to the previous case, $m+n-1 \geq m$ and moreover $P_2(X_2), P_2(Y_2) \neq \varnothing$. Then $\mathrm{m}_2 = \min\{m+n-1, m, n\} = m$. □

Proposition 5.1.15. *The simplicial complexes $\mathcal{H}^{(2)}_{1,n}$ and $\mathcal{H}^{(1)}_{1,n+1}$ are isomorphic, for every $n \in \mathbb{N}$ with $n \geq 2$. There are no other isomorphic simplicial complexes in the set*

$$\left\{\mathcal{U}_{3,5}, \mathcal{H}^{(1)}_{m,n}, \mathcal{H}^{(2)}_{m,n} \mid m, n \in \mathbb{N} \text{ with } m \leq n \text{ and } m+n \geq 3\right\}$$

Proof. Use the notation of Definition 5.1.11.

Let $n \in \mathbb{N}$. The number of points of the simplicial complexes $\mathcal{H}^{(2)}_{1,n}$ and $\mathcal{H}^{(1)}_{1,n+1}$ is clearly the same and equal to $n+2$. Moreover,

$$\begin{aligned} H^{(2)}_{1,n} &= P_3([n+2]) \setminus (P_3(\{1, n+2\}) \cup P_3(\{2, \dots, n+2\})) \\ &= P_3([n+2]) \setminus P_3(\{2, \dots, n+2\}) \\ &= P_3([n+2]) \setminus (P_3(\{1\}) \cup P_3(\{2, \dots, n+2\})) = H^{(1)}_{1,n+1} \end{aligned}$$

Suppose that there exist $m, n \in \mathbb{N}$ such that $\mathcal{U}_{3,5} \cong \mathcal{H}^{(1)}_{m,n}$. Then $|V_1| = 5$ and $P_{\leq 3}(V_1) \setminus Z_1 = H^{(1)}_{m,n} = P_{\leq 3}(V_1)$ and so $Z_1 = P_3(X_1) \cup P_3(Y_1) = \varnothing$. Therefore $|X_1|, |Y_1| \leq 2$ and so $|V_1| = |X_1| + |Y_1| \leq 4$, a contradiction.

Suppose that there exist $m, n \in \mathbb{N}$ such that $\mathcal{U}_{3,5} \cong \mathcal{H}^{(2)}_{m,n}$. Then $|V_2| = 5$ and $P_{\leq 3}(V_1) \setminus Z_2 = H^{(2)}_{m,n} = P_{\leq 3}(V_2)$ and so $Z_2 = P_3(X_2) \cup P_3(Y_2) = \varnothing$. Therefore $|X_2|, |Y_2| \leq 2$ and so $|V_2| = |X_2| + |Y_2| - 1 \leq 3$, a contradiction.

Suppose that there exist $m, n, k, \ell \in \mathbb{N}$ with $m \leq n$ and $k \leq \ell$ such that $\mathcal{H}^{(1)}_{m,n} \cong \mathcal{H}^{(1)}_{k,\ell}$. Then $m+n = k+\ell$ (because the number of points must be the same); by Lemma 5.1.14, $m = \min\{m,n\} = \min\{k,\ell\} = k$; and so we also get $n = \ell$.

Suppose that there exist $m, n, k, \ell \in \mathbb{N}$ with $m \leq n$ and $k \leq \ell$ such that $\mathcal{H}^{(2)}_{m,n} \cong \mathcal{H}^{(2)}_{k,\ell}$. Then $m+n+1 = k+\ell+1$ (because the number of points must be the same); by Lemma 5.1.14, $m = \min\{m,n\} = \min\{k,\ell\} = k$; and so we also get $n = \ell$.

Suppose that there exist $m, n, k, \ell \in \mathbb{N}$ with $m \leq n$ and $k \leq \ell$ such that $\mathcal{H}^{(1)}_{m,n} \cong \mathcal{H}^{(2)}_{k,\ell}$. Then $m+n = k+\ell+1$ (because the number of points must be the same); by Lemma 5.1.14, $m = \min\{m,n\} = \min\{k,\ell\} = k$; and so we also get $n = \ell+1$. Then the number of elements of the sets $P_3(V_1) \setminus H^{(1)}_{k,\ell+1}$ and $P_3(V_2) \setminus H^{(2)}_{k,\ell}$ must be the same, i.e. the following equalities hold:

$$\begin{aligned} &\binom{k}{3} + \binom{\ell+1}{3} = \binom{k+1}{3} + \binom{\ell+1}{3} \\ \iff &k(k-1)(k-2) = (k+1)k(k-1) \\ \iff &3k(k-1) = 0 \end{aligned}$$

We get a contradiction if $k \neq 1$. □

Definition 5.1.16. Define 1-complete boolean matrices M_0 and N_0 with columns indexed by $[5]$ such that

- $\mathcal{L}_{M_0} := \{12, 15, 23, 34, 45\}$
- $\mathcal{L}_{N_0} := \{13, 14, 25, 24, 35\}$

Moreover, given $m,n \in \mathbb{N}$ with $m \leq n$ and $m+n \geq 3$, define 1-complete boolean matrices $M^{(1)}_{m,n}$ and $N^{(1)}_{m,n}$ with columns indexed by $[m+n]$; $M^{(2)}_{m,n}$ and $N^{(2)}_{m,n}$ with columns indexed by $[m+n+1]$ such that:

- $\mathcal{L}_{M^{(1)}_{m,n}} := \{ab \mid a \in [m], b \in [m+n] \setminus [m]\}$
- $\mathcal{L}_{N^{(1)}_{m,n}} := \{[m], [m+n] \setminus [m]\}$
- $\mathcal{L}_{M^{(2)}_{m,n}} := \{ab \mid a \in [m], b \in [m+n] \setminus [m]\}$
- $\mathcal{L}_{N^{(2)}_{m,n}} := \{[m] \cup \{m+n+1\}, [m+n+1] \setminus [m]\}$

Proposition 5.1.17. *Given $m,n \in \mathbb{N}$ with $m \leq n$ and $m+n \geq 3$, the following properties hold:*

- *M_0, N_0 are boolean representations of $\mathcal{U}_{3,5}$*
- *$M^{(1)}_{m,n}$ and $N^{(1)}_{m,n}$ are boolean representations of $\mathcal{H}^{(1)}_{m,n}$*
- *$M^{(2)}_{m,n}$ and $N^{(2)}_{m,n}$ are boolean representations of $\mathcal{H}^{(2)}_{m,n}$*

Moreover,

$$\mathcal{L}_{M_0} \cap \mathcal{L}_{N_0} = \mathcal{L}_{M^{(1)}_{m,n}} \cap \mathcal{L}_{N^{(1)}_{m,n}} = \mathcal{L}_{M^{(2)}_{m,n}} \cap \mathcal{L}_{N^{(2)}_{m,n}} = \varnothing$$

and $M_0, M^{(1)}_{m,n}, M^{(2)}_{m,n}$ are short matrices.

Proof. Observe that the number of columns of the matrices and the number of points of the simplicial complexes coincide in all cases. Moreover, every set of lines defined above is a PEG, so the respective 1-complete boolean matrix represents a simplicial complex in BPav_2 by Proposition 4.1.5. Therefore we may use Proposition 4.1.6 and get:

$$\{\text{3-anticliques of } \mathscr{G}(M_0)\} = \{\text{3-anticliques of } \mathscr{G}(N_0)\} = \varnothing$$

$$\bigcup_{L \in \mathcal{L}_{M_0}} P_3(L) = \bigcup_{L \in \mathcal{L}_{N_0}} P_3(L) = \varnothing$$

$$H_{M_0} = H_{N_0} = P_{\leq 3}([5])$$

$$\left\{\text{3-anticliques of } \mathscr{G}(M^{(1)}_{m,n})\right\} = \bigcup_{L \in \mathcal{L}_{N^{(1)}_{m,n}}} P_3(L) = P_3([m]) \cup P_3([m+n] \setminus [m])$$

$$\left\{\text{3-anticliques of } \mathscr{G}(N^{(1)}_{m,n})\right\} = \bigcup_{L \in \mathcal{L}_{M^{(1)}_{m,n}}} P_3(L) = \varnothing$$

$$H_{M^{(1)}_{m,n}} = H_{N^{(1)}_{m,n}} = P_{\leq 3}([m+n]) \setminus (P_3([m]) \cup P_3([m+n] \setminus [m]))$$

$$\left\{\text{3-anticliques of } \mathscr{G}(M^{(2)}_{m,n})\right\} = \bigcup_{L \in \mathcal{L}_{N^{(2)}_{m,n}}} P_3(L) = P_3([m] \cup \{m+n+1\}) \cup P_3([m+n+1] \setminus [m])$$

$$\left\{\text{3-anticliques of } \mathscr{G}(N^{(2)}_{m,n})\right\} = \bigcup_{L \in \mathcal{L}_{M^{(2)}_{m,n}}} P_3(L) = \varnothing$$

$$H_{M^{(2)}_{m,n}} = H_{N^{(2)}_{m,n}} = P_{\leq 3}([m+n]) \setminus (P_3([m] \cup \{m+n+1\}) \cup P_3([m+n+1] \setminus [m]))$$

The last assertion follows clearly. $\square$

5.2 General Case

Context 5.2.1. We start this Section by establishing some needed definitions and results. So, up until Lemma 5.2.6, let $\mathcal{H} = (V, H) \in \mathrm{BPav}_2$ and $M, N \in \mathrm{BR}(\mathcal{H})$ such that $\mathcal{L}_M \cap \mathcal{L}_N = \varnothing$.

Definition 5.2.2. Define:

- The set $E_{M,N} := \left(\bigcup_{L\in\mathcal{L}_M} P_2(L)\right) \cup \left(\bigcup_{J\in\mathcal{L}_M} P_2(J)\right)$
- The graph $\Gamma_{M,N} := (V, E_{M,N})$
- The map $\begin{array}{rccl} c_{M,N}: & E_{M,N} & \to & [2] \\ & X & \mapsto & \begin{cases} 1 & \text{if } X \in \bigcup_{L\in\mathcal{L}_M} P_2(L) \\ 2 & \text{if } X \in \bigcup_{J\in\mathcal{L}_N} P_2(J) \end{cases} \end{array}$

Remark 5.2.3. The map $c_{M,N}$ is a well-defined edge-colouring of $\Gamma_{M,N}$.

Proof. Suppose that there exists an edge $X \in E_{M,N}$ and lines $L \in \mathcal{L}_M$ and $J \in \mathcal{L}_N$ such that $X \subseteq L \cap J$. Observe that $L \neq J$ because $\mathcal{L}_M \cap \mathcal{L}_N = \varnothing$. Therefore $|L \cap J| \geq |X| = 2$, contradicting Proposition 4.1.2. Hence, $c_{M,N}(X)$ is uniquely determined for every edge $X \in E_{M,N}$. □

Definition 5.2.4. Define the edge-coloured graph $\mathrm{g}_{M,N} := (\Gamma_{M,N}, c_{M,N})$.

Note 5.2.5. Throughout this Section, we use colours blue and red refering to 1 and 2, respectively. We use both terminology as convenient.

Lemma 5.2.6. *Given a subset $V' \subseteq V$, define the submatrices $M' := M[-, V']$ and $N' := N[-, V']$. Then M' and N' are boolean representations of $\mathcal{H}|_{V'}$ and $\mathcal{L}_{M'} \cap \mathcal{L}_{N'} = \varnothing$. Moreover, one of the following conditions holds:*

- $\mathcal{L}_{M'} = \varnothing$
- $\mathcal{L}_{N'} = \varnothing$
- $\mathrm{g}_{M,N}[V'] = \mathrm{g}_{M',N'}$

Proof. The first assertion follows clearly from Proposition 3.4.10, Remark 3.4.11 and Proposition 4.1.2. Also by Remark 3.4.11 we may conculde that:

- $\bigcup_{L\in\mathcal{L}_{M'}} P_2(L) \subseteq \bigcup_{L\in\mathcal{L}_M} P_2(L)$
- $\bigcup_{J\in\mathcal{L}_{N'}} P_2(J) \subseteq \bigcup_{J\in\mathcal{L}_N} P_2(J)$

Hence $E_{M',N'} \subseteq E_{M,N} \cap 2^{V'}$ and ${c_{M',N'}}^{-1}(i) \subseteq {c_{M,N}|_{E_{M,N\cap 2^{V'}}}}^{-1}(i)$ for every colour $i \in [2]$. Conversely, suppose that $\mathcal{L}_{M'}, \mathcal{L}_{N'} \neq \varnothing$ and consider sets $X \in \bigcup_{L\in\mathcal{L}_{M'}} P_2(L)$ and $Y \in \bigcup_{J\in\mathcal{L}_{N'}} P_2(J)$. Therefore there exist lines $L_0 \in \mathcal{L}_M$ and $J_0 \in \mathcal{L}_N$ such that $X \in P_2(L_0) \cap 2^{V'}$ and $Y \in P_2(J_0) \cap 2^{V'}$. We claim that $L_0 \cap V' \in \mathcal{L}_{M'}$ and $J_0 \cap V' \in \mathcal{L}_{N'}$. To prove the claim observe that $X \subseteq L_0 \cap V'$ and $Y \subseteq J_0 \cap V'$. So $|L_0 \cap V'|, |J_0 \cap V'| \geq 2$. Moreover, suppose that $L_0 \cap V' = V'$ (or $J_0 \cap V' = V'$). Then all the other rows in M' (or N') contain at most one entry equal to 0 by Proposition 4.1.2, because $\mathcal{H} \in \mathrm{BPav}_2$. Therefore $\mathcal{L}_{M'} = \varnothing$ (or $\mathcal{L}_{N'} = \varnothing$), contradicting the initial assumption. Hence the claim holds and we may conclude that:

- $\bigcup_{L\in\mathcal{L}_M} P_2(L) \subseteq \bigcup_{L\in\mathcal{L}_{M'}} P_2(L)$
- $\bigcup_{J\in\mathcal{L}_N} P_2(J) \subseteq \bigcup_{J\in\mathcal{L}_{N'}} P_2(J)$

Hence $E_{M,N} \cap 2^{V'} \subseteq E_{M',N'}$ and $c_{M,N}|_{E_{M,N\cap 2^{V'}}}{}^{-1}(i) \subseteq c_{M',N'}{}^{-1}(i)$ for every colour $i \in [2]$. So the equalities hold and $\mathrm{g}_{M,N}[V'] = \mathrm{g}_{M',N'}$. □

Context 5.2.7. Throughout this Section, we aim at getting a full description of simplicial complexes in BPav_2 admitting alternative 1-complete boolean representations with disjoint sets of lines, none of them being short. So, up until Proposition 5.2.37, let $\mathcal{H} = (V, H) \in \mathrm{BPav}_2$, consider $M, N \in \mathrm{BR}_{\text{1-complete}}(\mathcal{H}) \setminus \mathrm{BR}_{\text{short}}(\mathcal{H})$ such that $\mathcal{L}_M \cap \mathcal{L}_N = \varnothing$ and use E, Γ, c and g to denote the set of edges $E_{M,N}$, the graph $\Gamma_{M,N}$, the edge-colouring $c_{M,N}$ and the edge-coloured graph $\mathrm{g}_{M,N}$, respectively.

Note 5.2.8. For technical purposes, define the map $\overline{c} : E \to [2]$ such that $\overline{c}(X) = 3 - c(X)$ for every edge $X \in E$. Then $\overline{c}$ is clearly an edge-colouring of Γ. Define the edge-coloured graph $\overline{\mathrm{g}} := (\Gamma, \overline{c})$. Say that $\overline{\mathrm{g}}$ is the *colour inversion* of g.

Remark 5.2.9. g contains a red and a blue triangle. Therefore $|V| \geq 5$.

Proof. None of the representations M and N is short. So, there exist lines $L \in \mathcal{L}_M$ and $J \in \mathcal{L}_N$ such that $|L|, |J| \geq 3$. Then $\mathrm{g}[L]$ is a blue triangle and $\mathrm{g}[J]$ is a red triangle. Moreover, $|L \cap J| \leq 1$ and therefore $|V| \geq |L| + |J| - |L \cap J| \geq 3 + 3 - 1 \geq 5$. □

Lemma 5.2.10. *Given $Z \in P_3(V)$, one of the following conditions holds:*

1. *Z is an anticlique of Γ*
2. *$\mathrm{g}[Z]$ is a monochromatic triangle*
3. *$\mathrm{g}[Z]$ contains edges of different colours*

Moreover, $Z \in H$ if and only if condition 3 holds.

Proof. Recall that $\mathcal{H} \in \mathrm{BPav}_2$ and $M, N \in \mathrm{BR}(\mathcal{H})$. So we may use both descriptions of $H = H_M = H_N$ in Proposition 4.1.6. Using the first equality in Proposition 4.1.6, the following statements are equivalent (sequential equivalences are clear):

- $Z \in H = H_M = H_N$
- $Z \in \bigcup_{L\in\mathcal{L}_M} \{X \in P_3(V) \mid |X \cap L| = 2\}$ and $Z \in \bigcup_{J\in\mathcal{L}_N} \{Y \in P_3(V) \mid |Y \cap J| = 2\}$
- There exist lines $L_0 \in \mathcal{L}_M$ and $J_0 \in \mathcal{L}_N$ such that $|Z \cap L_0| = |Z \cap J_0| = 2$
- $\mathrm{g}[Z]$ contains edges of different colours

Now, using the second equality in Proposition 4.1.6, the following statements are equivalent:

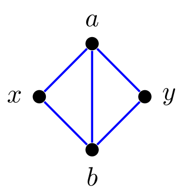
1. $Z \notin H = H_M = H_N$
2. $Z \in \{$3-anticliques of $\mathscr{G}(M)\} \cup \left(\bigcup_{L\in\mathcal{L}_M} P_3(L)\right)$ and
   $Z \in \{$3-anticliques of $\mathscr{G}(N)\} \cup \left(\bigcup_{J\in\mathcal{L}_N} P_3(J)\right)$
3. $Z \in \{$3-anticliques of $\mathscr{G}(M)\} \cap \{$3-anticliques of $\mathscr{G}(N)\}$ or
   $Z \in \{$3-anticliques of $\mathscr{G}(M)\} \cap \left(\bigcup_{J\in\mathcal{L}_N} P_3(J)\right)$ or
   $Z \in \{$3-anticliques of $\mathscr{G}(N)\} \cap \left(\bigcup_{L\in\mathcal{L}_M} P_3(L)\right)$

4. Z is an anticlique of Γ or $\mathrm{g}\,[Z]$ is a monochromatic triangle

Statements 2 and 3 are equivalent because $\left(\bigcup_{L\in\mathcal{L}_M} P_3(L)\right) \cap \left(\bigcup_{J\in\mathcal{L}_N} P_3(J)\right) = \varnothing$ by Proposition 4.1.2 and because $\mathcal{L}_M \cap \mathcal{L}_N = \varnothing$. The remaining sequential equivalences are clear. $\square$

Lemma 5.2.11. *If* g *contains two monochromatic triangles sharing one edge, then the edge-coloured subgraph induced by their vertices is a monochromatic* K_4.

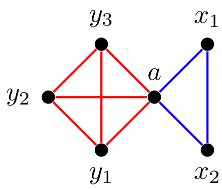
Proof. Without loss of generality, suppose that g contains blue triangles with vertex sets abx and aby, respectively ($a, b, x, y \in V$ are distinct vertices). We get the following configuration:



There exists a line $L \in \mathcal{L}_M$ such that $ab \subseteq L$. By Proposition 5.2.10, $abx, aby \notin H$ and therefore $x, y \in L$. So $abxy \subseteq L$ and then $\mathrm{g}\,[abxy]$ is a blue K_4. $\square$

Lemma 5.2.12. *If* g *contains a monochromatic triangle and a monochromatic* K_4 *of different colours, then they share no vertex.*

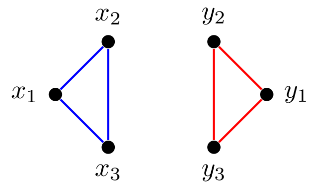
Proof. Without loss of generality, suppose that g contains a blue triangle and a red K_4 with vertex sets $X := ax_1x_2$ and $Y := ay_1y_2y_3$, respectively ($a, x_1, x_2, y_1, y_2, y_3 \in V$ are pairwise distinct vertices). We get the following configuration:



We claim that, for every index $i \in \{1, 2, 3\}$, one of the sets x_1y_i or x_2y_i is a red edge of g. To prove the claim, observe that $x_1x_2y_i$ is not an anticlique of Γ (because x_1x_2 is an edge) and, by Lemma 5.2.11, $\mathrm{g}\,[x_1x_2y_i]$ is not a blue triangle (because $\mathrm{g}\,[X \cup \{y_i\}]$ is not a blue K_4). Therefore, by Proposition 5.2.10, $\mathrm{g}\,[x_1x_2y_i]$ contains a red edge and the claim holds. Hence, we may conclude that there are at least 3 red edges joining $Y \setminus \{a\}$ and $X \setminus \{a\}$. So there exist a vertex $x \in X \setminus \{a\}$ such that there are at least 2 red edges joining x and $Y \setminus \{a\}$. Consider then distinct vertices $y, y' \in Y \setminus \{a\}$ such that $\mathrm{g}\,[xyy']$ is a red triangle. Then, by Lemma 5.2.11, $\mathrm{g}\,[xyy'a]$ is a red K_4, a contradiction (because ax is a blue edge). $\square$

Lemma 5.2.13. *Every pair of monochromatic triangles of different colours in* g *share one vertex.*

Proof. Suppose that g contains a blue and a red triangle of vertex sets $X = x_1x_2x_3$ and $Y = y_1y_2y_3$, respectively, with $X \cap Y = \varnothing$. We get the following configuration:



We claim that, for all indices $i, j, k \in \{1, 2, 3\}$ with $j \neq k$, one of the sets $x_i y_j$ or $x_i y_k$ is a blue edge of g. To prove the claim, observe that $x_i y_j y_k$ is not an anticlique of Γ (because $y_j y_k$ is an edge) and, by Lemma 5.2.12, $\mathrm{g}[Y \cup \{x_i\}]$ is not a red K_4. So, by Lemma 5.2.11, $\mathrm{g}[x_i y_j y_k]$ is not a red triangle and therefore, by Proposition 5.2.10, $\mathrm{g}[x_i y_j y_k]$ contains a blue edge and the claim holds. Hence, we may conclude that there are at least 6 blue edges joining X and Y. So there exist a vertex $y \in Y$ such that there are at least 2 blue edges joining y and X. Consider then distinct vertices $x, x' \in X$ such that $\mathrm{g}[xx'y]$ is a blue triangle. Then, by Lemma 5.2.11, $\mathrm{g}[X \cup \{y\}]$ is a blue K_4 that shares one vertex with the red triangle $\mathrm{g}[X]$, contradicting Lemma 5.2.12. □

Definition 5.2.14. Define (up to graph isomorphism) the graph K_4' obtained from the complete graph K_4 by removing one edge.

Proposition 5.2.15. g *does not contain a monochromatic* K_4'*.*

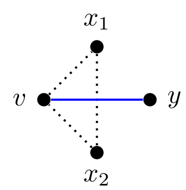
Proof. Without loss of generality, suppose that g contains a blue K_4' with vertex set X. Observe that g contains a red triangle because N is not a short representation of $\mathcal{H}$. Let Y be the vertex set of a red triangle in g. Moreover, by Lemma 5.2.11, $\mathrm{g}[X]$ is a blue K_4 and therefore $|X \cap Y| \leq 1$. So there exists a subset $X' \in P_3(X) \setminus Y$. Hence $\mathrm{g}[X']$ and $\mathrm{g}[Y]$ are monochromatic triangles of different colours sharing no vertex, contradicting Lemma 5.2.13. □

Lemma 5.2.16. *There are no vertices of degree 0 in* Γ*.*

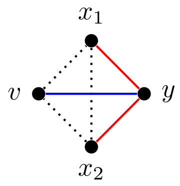
Proof. Recall that $\dim \mathcal{H} = 2$. So $|V| \geq 3$; and there exists a set $X \in P_3(V) \cap H$. Then, by Proposition 5.2.10, $\mathrm{g}[X]$ contains edges of different colours and so $E \neq \varnothing$. Suppose now that there exists a vertex $v \in V$ such that $\deg_\Gamma(v) = 0$. Then, by Proposition 5.2.10, for every distinct vertices $x, y \in V \setminus \{v\}$, xyv is an anticlique of Γ. Hence $E = \varnothing$, a contradiction. □

Lemma 5.2.17. $\deg_\Gamma(v) \geq |V| - 2$ *for every vertex* $v \in V$*.*

Proof. Consider a vertex $v \in V$. Suppose that there exist distinct vertices $x_1, x_2 \in V \setminus \{v\}$ such that $vx_1, vx_2 \notin E$. Then, by Proposition 5.2.10, vx_1x_2 is an anticlique of Γ. By Lemma 5.2.16, there exists a vertex $y \in V \setminus vx_1x_2$ such that $vy \in E$ (assume that vy is a blue edge). We get the following configuration:



Then, by Proposition 5.2.10, x_1y and x_2y are red edges and we get the following configuration:



Also by Proposition 5.2.10, we may conclude that $x_1x_2 \in E$, a contradiction. $\square$

5.2.1 Non Complete Case

Context 5.2.18. Throughout this Subsection, assume that Γ is not a complete graph. Let $v_1, v_2 \in V$ be distinct vertices such that $v_1v_2 \notin E$.

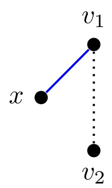
Remark 5.2.19. $\mathcal{N}_\Gamma(v_1) = \mathcal{N}_\Gamma(v_2) = V \setminus v_1v_2$.

Proof. Follows clearly from Lemma 5.2.17. $\square$

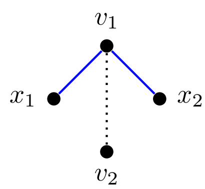
Lemma 5.2.20. *The following conditions hold:*

1. *Given a colour* $i \in [2]$, $\mathcal{N}^{(i)}(v_1) = \mathcal{N}^{(3-i)}(v_2)$.
2. *Given a colour* $i \in [2]$ *and a vertex* $v \in v_1v_2$, $P_2(\mathcal{N}^{(i)}(v)) \subseteq E$.

Proof. Consider a colour $i \in [2]$ and an index $p \in [2]$. Consider a vertex $x \in \mathcal{N}^{(i)}(v_p)$. We claim that $x \in \mathcal{N}^{(3-i)}(v_{3-p})$. Assume, without loss of generality, that $i = p = 1$. We get the following configuration:

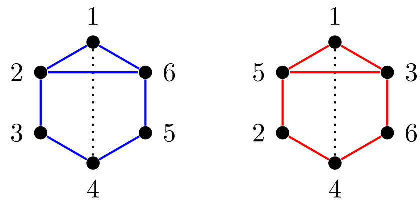


Then, by Proposition 5.2.10, xv_2 is a red edge and so $x \in \mathcal{N}^{(2)}(v_2)$. Hence the claim holds and therefore $\mathcal{N}^{(i)}(v_p) \subseteq \mathcal{N}^{(3-i)}(v_{3-i})$ for all colours $i \in [2]$ and indices $p \in [2]$. So condition 1 holds. Consider now an element $v \in v_1v_2$ and a set $X \in P_2(\mathcal{N}^{(i)})(v)$. We claim that $X \in E$. Assume, without loss of generality, that $i = 1$ and $v = v_1$ and write $X = x_1x_2$. We get the following configuration:



Then, by Proposition 5.2.10, $X \in E$ and so condition 2 holds. $\square$

Definition 5.2.21. Define the edge-coloured graph $g_0 = (\Gamma_0, c_0)$ such that $\Gamma_0 := ([6], P_2([6]) \setminus \{14\})$; c_0 is an edge-colouring of Γ_0 with 2 colours; and $\Gamma_0^{(1)}, \Gamma_0^{(2)}$ are the following graphs, respectively:

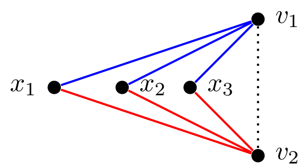


Lemma 5.2.22.

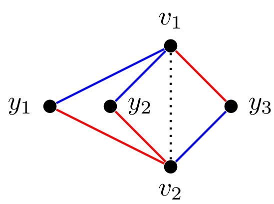
1. Given a colour $i \in [2]$ and a vertex $v \in v_1v_2$, $\deg^{(i)}(v) \leq 2$.

2. $|V| = 6$.

Proof. Suppose that there exist a colour $i \in [2]$ and a vertex $v \in v_1v_2$ such that $\deg^{(i)}(v) \geq 3$. Assume, without loss of generality, that $i = 1$ and $v = v_1$ and consider distinct vertices $x_1, x_2, x_3 \in \mathcal{N}^{(1)}(v_1)$. Using Lemma 5.2.20 (1), we get the following:



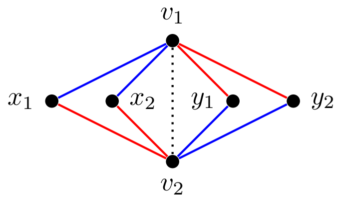
Now, by Lemma 5.2.20 (2), $x_1x_2, x_1x_3, x_2x_3 \in E$ and at least two of these edges have the same colour $j \in [2]$. So $\mathrm{g}\,[x_1x_2x_3v_j]$ contains a monochromatic K_4' contradicting Proposition 5.2.15. Hence property 1 holds. Recall that $|V| \geq 5$ by Remark 5.2.9. Consider now a vertex $v \in V$ and observe that $\deg_\Gamma(v) = \deg^{(1)}(v) + \deg^{(2)}(v)$. So $\deg_\Gamma(v) \leq 4$ by property 1. Moreover, by Lemma 5.2.17, $|V| \leq \deg_\Gamma(v) + 2$ and therefore $|V| \leq 6$. Suppose that $|V| = 5$. Write $V = v_1v_2y_1y_2y_3$ ($y_1, y_2, y_3 \in V \setminus v_1v_2$ are distinct vertices). By property 1, we may assume without loss of generality that $c(v_1y_1) = c(v_1y_2) = 1$ and $c(v_1y_3) = 2$. Using Lemma 5.2.20 (1), we get the following:



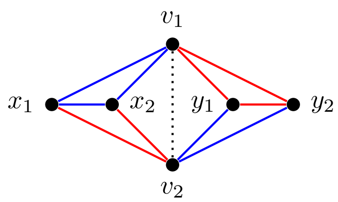
By Lemma 5.2.20 (2), $y_1y_2 \in E$. Observe now that, for every colour $p \in [2]$, if $c(y_1y_2) = p$ then g does not contain a triangle of colour $3 - p$, contradicting Remark 5.2.9. Hence we may conclude that $|V| = 6$. □

Corollary 5.2.23. *g and g_0 are isomorphic edge-coloured graphs.*

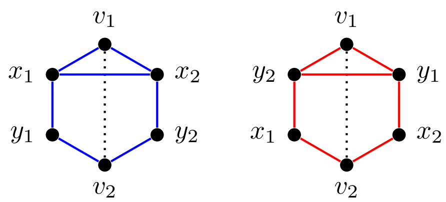
Proof. Write $V = v_1v_2x_1x_2y_1y_2$. By Lemma 5.2.22, we may assume that the vertices $x_1, x_2, y_1, y_2 \in V \setminus v_1v_2$ are distinct, $c(v_1x_p) = 1$ and $c(v_1y_q) = 2$ for all indices $p, q \in [2]$. By Lemma 5.2.20 (1), we get the following:



By Lemma 5.2.20 (2), $x_1x_2, y_1y_2 \in E$. Observe that, for all indices $i, j, k \in [2]$, $\mathrm{g}\,[x_iy_jv_k]$ contains edges of different colours. So every monochromatic triangle in g contains one of the edges x_1x_2, y_1y_2. But then the edges x_1x_2, y_1y_2 have different colours (because g contains a blue and a red triangle by Remark 5.2.9). We may assume that $c(x_1x_2) = 1$ and $c(y_1y_2) = 2$ because the edge-coloured graph above and its colour inversion are isomorphic. We get the following:



By Lemma 5.2.10 and Proposition 5.2.15, for all indices $i, j \in [2]$, one of the edges y_ix_1, y_ix_2 is red; and one of the edges x_jy_1, x_jy_2 is blue. Thus there exist two blue edges and two red edges among the four edges y_ix_j. Then we may assume that $c(x_1y_1) = c(x_2y_2) = 1$ and $c(x_1y_2) = c(x_2y_1) = 2$ because the permutation $(x_1x_2)(y_1y_2)$ is an automorphism of the graph above. We get the following graphs $\Gamma^{(1)}$ and $\Gamma^{(2)}$, respectively:

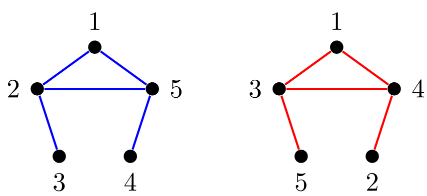


$\square$

5.2.2 Complete Case

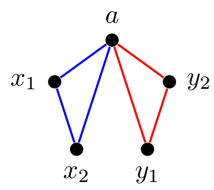
Context 5.2.24. Throughout this Subsection we assume that Γ is a complete graph.

Definition 5.2.25. Define the edge-coloured graph $\mathrm{g}_1 = (\Gamma_1, c_1)$ such that $\Gamma_1 := ([5], P_2([5]))$; c_1 is an edge-colouring of Γ_1 with 2 colours; and $\Gamma_1^{(1)}, \Gamma_1^{(2)}$ are the following graphs, respectively:

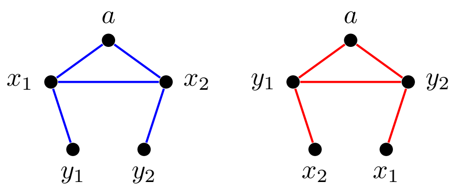


Proposition 5.2.26. $|V| = 5$ *if and only if* g *and* g_1 *are isomorphic edge-coloured graphs.*

Proof. By Remark 5.2.9 there exist a blue and a red triangle in g. By Lemma 5.2.13, every pair of triangles of different colours in g share a vertex. So, consider a blue and a red triangle in g with vertex sets $X = ax_1x_2$ and $Y = ay_1y_2$ ($a, x_1, x_2, y_1, y_2 \in V$ are distinct vertices, so $V = ax_1x_2y_1y_2$). We get the following:



By Proposition 5.2.15, for all indices $i, j \in [2]$, one of the edges y_ix_1, y_ix_2 is red; and one of the edges x_jy_1, x_jy_2 is blue. Then we may assume that $c(x_1y_1) = c(x_2y_2) = 1$ and $c(x_1y_2) = c(x_2y_1) = 2$ because the permutation $(x_1x_2)(y_1y_2)$ is an automorphism of the graph above. We get the following graphs $\Gamma^{(1)}$ and $\Gamma^{(2)}$, respectively:



□

Lemma 5.2.27. *If* $|V| \neq 5$*, given a subset* $X \in P_3(V)$ *such that* $\mathrm{g}[X]$ *is a monochromatic triangle, then* $\dim\ \mathcal{H}|_{V\setminus X} = 2$*.*

Proof. Write $X = x_1x_2x_3$, $V' := V \setminus X$ and $\mathcal{H}' := \mathcal{H}|_{V'} =: (V', H')$. Observe that $V' \neq \varnothing$ by Remark 5.2.9. Moreover, $\dim \mathcal{H}' \leq \dim \mathcal{H} = 2$. Suppose first that $\dim \mathcal{H}' = 0$ and so $H' = P_1(V')$. We claim that $|V'| = 1$. To prove the claim, suppose that there exist distinct vertices $x, y \in V'$. Then $xy \in P_2(V)$ and so $xy \in H$ because $P_2(V) \subseteq H$ (i.e. $\mathcal{H}$ is paving). Therefore $xy \in H' = 2^{V'} \cap H$, a contradiction. Hence the claim holds and so $|V| = |V'| + 3 = 4$, contradicting Remark 5.2.9.

Suppose now that $\dim \mathcal{H}' = 1$. If $|V'| = 2$ then $|V| = |V'| + 3 = 5$. We may then assume that $|V'| \geq 3$. We claim that all the edges in $\mathrm{g}[V']$ have the same colour. To prove the claim, suppose that there exist distinct vertices $y, z_1, z_2 \in V'$ such that yz_1 and yz_2 are edges of different colours. Then, by Proposition 5.2.10, $yz_1z_2 \in H$ and therefore $yz_1z_2 \in H' = H \cap 2^{V'}$, a contradiction (because $\dim \mathcal{H}' = 1$). So the claim holds. Recall that Γ is the complete graph with vertex set V and assume without loss of generality that $\mathrm{g}[X]$ is a blue triangle. Now, if all the edges in $\mathrm{g}[V']$ are blue, then there are no red triangles in g, contradicting Remark 5.2.9; if all the edges in $\mathrm{g}[V']$ are red, then there exists in g a pair of monochromatic triangles of different colours sharing no vertex, contradicting Lemma 5.2.13. Hence we may conclude that $\dim \mathcal{H}' = 2$. □

Remark 5.2.28. Given a subset $X \in P_3(V)$ such that $\mathrm{g}[X]$ is a monochromatic triangle, define the set $V' := V \setminus X$, the simplicial complex $\mathcal{H}' := \mathcal{H}|_{V'}$ and the submatrices $M' := M[-, V']$

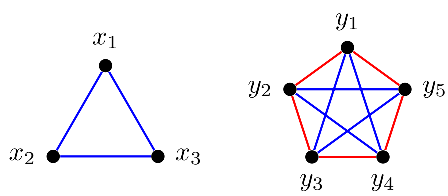
and $N' := N[-, V']$. If $|V| \neq 5$, then $M', N' \in \mathrm{BR}_{\text{1-complete}}(\mathcal{H}')$ and $\mathcal{L}_{M'} \cap \mathcal{L}_{N'} = \varnothing$. Moreover, $\mathrm{g}[V'] = \mathrm{g}_{M',N'}$.

Proof. By Lemma 5.2.6, $M', N' \in \mathrm{BR}(\mathcal{H}')$. Moreover, M, N are 1-complete matrices and the submatrices M', N' are obtained from M', N' by removing columns. So clearly $M', N' \in \mathrm{BR}_{\text{1-complete}}(\mathcal{H}')$. By Lemma 5.2.27, $\dim \mathcal{H}' = 2$ and therefore, by Proposition 4.1.6, $\mathcal{L}_{M'}, \mathcal{L}_{N'} \neq \varnothing$. So the result follows now clearly from Lemma 5.2.6. □

Lemma 5.2.29. *Given a colour $i \in [2]$ and a subset $X \in P_3(V)$ such that $\mathrm{g}[X]$ is a monochromatic triangle of colour i, define the subgraph $\Gamma' := \Gamma^{(3-i)}[V \setminus X]$. If $|V| \neq 5$, then one of the following conditions holds:*

1. *$\Gamma' \in \Omega(m, n)$ for some $m, n \in \mathbb{N}$ such that $m + n = |V| - 3$*
2. *Γ' is isomorphic to the graph obtained from the complete bipartite graph $K_{p,q}$ by adding a trivial connected component, for some $p, q \in \mathbb{N}$ such that $s + t = |V| - 4$*

Proof. Write $V' := V \setminus X$ and assume without loss of generality that $i = 1$, so $\mathrm{g}[X]$ is a blue triangle. Then, by Lemma 5.2.13, $\mathrm{g}[V']$ does not contain red triangles. Therefore, by Remark 5.2.28, $\mathrm{g}_{M[-,V'],N[-,V']}$ does not contain red triangles. Hence, by Proposition 4.1.6, $N[-, V']$ is a short representation of $\mathcal{H}|_{V'}$ and, by Lemma 5.2.27, $\dim \mathcal{H}' = 2$. Then, by Propositions 5.1.9 and 5.1.10, Γ' and $\mathcal{C}_5$ are isomorphic graphs or one of the conditions 1 and 2 hold. Suppose that Γ' and $\mathcal{C}_5$ are isomorphic graphs. Write $X = x_1x_2x_3$ and $Y := V \setminus X = y_1 \dots y_5$ and get the following:



By Proposition 5.2.15, for every index $i \in [5]$, at least two of the edges y_ix_1, y_ix_2, y_ix_3 are red. Then there are at least 10 red edges joining Y and X. So, there exists a vertex $x \in X$ and a subset $Y' \in P_4(Y)$ such that $Y' \subseteq \mathcal{N}^{(2)}(x)$. But then $\mathrm{g}[Y' \cup \{x\}]$ contains a red K_4', contradicting Proposition 5.2.15. Therefore Γ' and $\mathcal{C}_5$ are not isomorphic graphs and one of the desired conditions holds. □

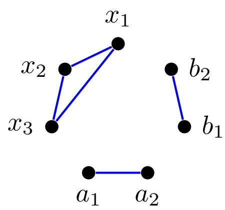
Lemma 5.2.30. $5 \leq |V| \leq 9$. *Moreover, g contains a pair of triangles of the same colour with no common vertices if and only if $|V| \geq 7$.*

Proof. By Remark 5.2.9, $|V| \geq 5$. Consider a colour $i \in [2]$ and suppose that g contains a pair of monochromatic triangles of colour i and vertex sets X and Y, respectively, with $X \cap Y = \varnothing$. Then $\Gamma^{(3-i)}[X \cup Y]$ is a triangle-free graph. But by Remark 5.2.9, there exists a red triangle in g. Then there exists a vertex $v \in V \setminus (X \cup Y)$ and therefore $|V| \geq |X| + |Y| + 1 = 7$.

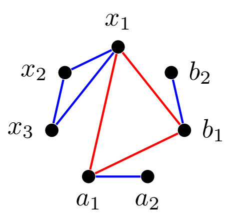
We must show now that $|V| \leq 9$ and that, if $|V| \geq 7$, then g contains a pair of triangles of the same colour with no common vertices. Consider a set $X \in P_3(V)$ such that $\mathrm{g}[X]$ is a monochromatic triangle of colour $i \in [2]$ (there exists such a subset by Remark 5.2.9). Assume without loss of generality that $i = 1$. We may also assume that g and g_1 are not isomorphic graphs (because otherwise the result holds clearly). We proceed now by analyzing the conditions presented in Lemma 5.2.29 separatedly.

Case 1:

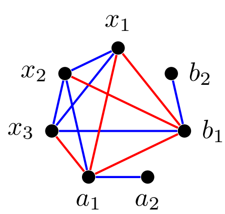
Suppose that condition 1 of Lemma 5.2.29 holds. Then we may consider a partition $V = X \dot\cup A \dot\cup B$ such that $\Gamma^{(2)}[A \cup B]$ can be obtained from the complete bipartite graph with partition sets A and B by removing a matching. Hence all the edges in the edge-coloured subgraphs $\mathrm{g}[A]$ and $\mathrm{g}[B]$ are blue and therefore, by Proposition 5.2.15, $|A|, |B| \leq 3$. So $|V| = |X| + |A| + |B| \leq 9$. Moreover, if $|V| \geq 8$ then one of the sets A and B contains exactly 3 elements. Assume without loss of generality that $|A| = 3$. Then $\mathrm{g}[X]$ and $\mathrm{g}[A]$ are a pair of blue triangles with no common vertices. Assume now that $|V| = 7$, write $X = x_1x_2x_3$, $A = a_1a_2$, $B = b_1b_2$ and get the following:



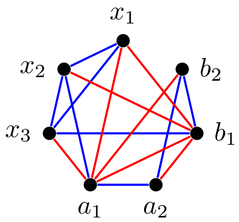
Every red triangle in g contains one vertex of each of the sets X, A and B. Therefore we may assume that $\mathrm{g}[x_1a_1b_1]$ is a red triangle (because, for every permutation $\sigma \in S_X$, $\sigma(a_1a_2)(b_1b_2)$ is an automorphism of the edge-coloured graph above) and get the following:



$\mathrm{g}[X \cup \{a_1, b_1\}]$ and g_1 are isomorphic graphs. Therefore, there exists an index $i \in \{2, 3\}$ such that $x_ia_1, x_{5-i}b_1$ are blue edges and $x_{5-i}a_1, x_ib_1$ are red edges. We may assume that $i = 2$ (because the permutation (x_2x_3) is an automorphism of the edge-coloured graph above) and get the following:



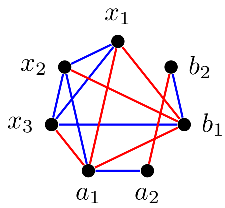
If $c(a_2b_2) = 1$ then $c(a_1b_2) = c(a_2b_1) = 2$ because $\Gamma^{(2)}[A \cup B] \in \Omega(2, 2)$. We get the following:



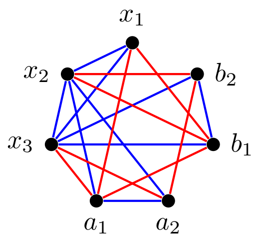
Therefore the following implications hold, by sequentially applying Proposition 5.2.15:

$$c(a_1b_2) = 2 \implies c(x_1b_2) = 1 \implies c(x_2b_2) = c(x_3b_2) = 2 \qquad \text{(so } \mathrm{g}\left[x_3a_1b_2\right] \text{ is a red triangle)}$$
$$c(a_2b_1) = 2 \implies c(x_1a_2) = 1 \implies c(x_2a_2) = c(x_3a_2) = 2 \qquad \text{(so } \mathrm{g}\left[x_2a_2b_1\right] \text{ is a red triangle)}$$

Therefore $\mathrm{g}\left[x_3a_1b_2\right]$ and $\mathrm{g}\left[x_2a_2b_1\right]$ are a pair of red triangles with no common vertices. On the other hand, if $c(a_2b_2) = 2$ we get:



If we assume that g does not contain a pair of red triangles with no common vertices, then, for every index $i \in \{2,3\}$, one of the edges x_ia_2, x_ib_2 is blue. Moreover, by Proposition 5.2.15, for every vertex $v \in a_2b_2$, one of the edges vx_2, vx_3 is red. Therefore we may assume that $c(x_2a_2) = c(x_3b_2) = 1$ and $c(x_2b_2) = c(x_3a_2) = 2$ (because the permutation $(x_2x_3)(a_1b_1)(a_2b_2)$ is an automorphism of the edge-coloured graph above) and get the following:

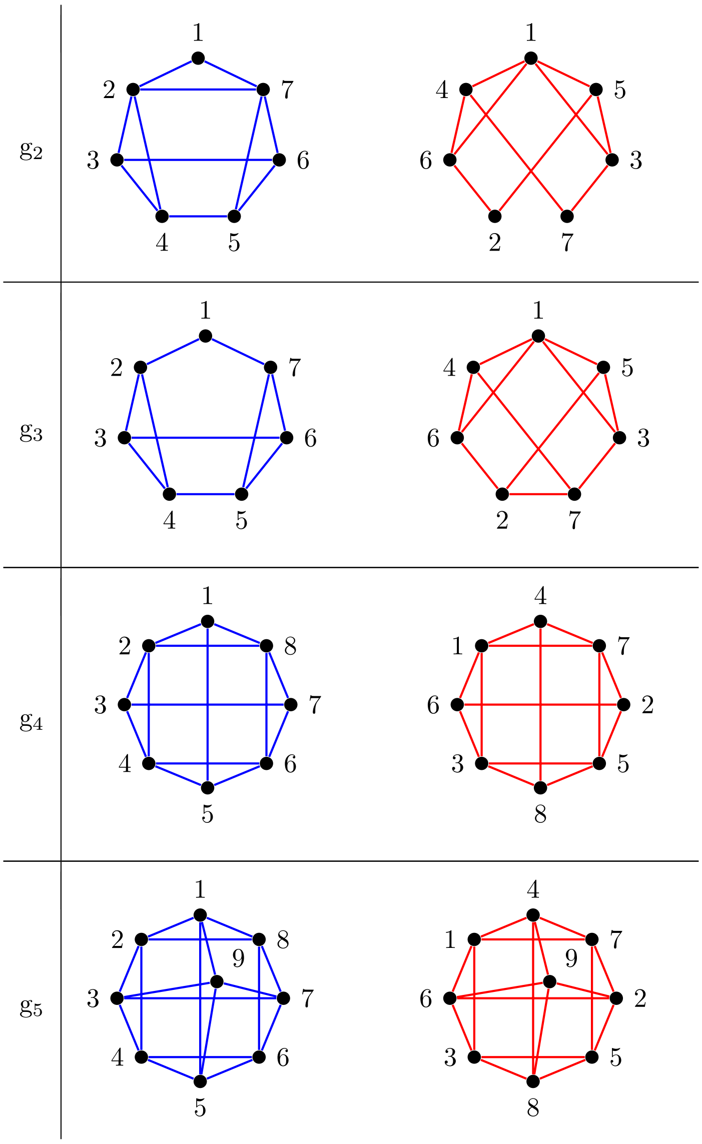


Then $\mathrm{g}\left[x_2a_1a_2\right]$ and $\mathrm{g}\left[x_3b_1b_2\right]$ are a pair of blue triangles with no common vertices.

Case 2:

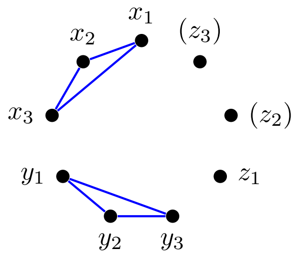
Suppose now that condition 2 of Lemma 5.2.29 holds. Then we may consider an element $v \in V$ and a partition $V = X \mathbin{\dot\cup} A \mathbin{\dot\cup} B \mathbin{\dot\cup} \{v\}$ such that $\Gamma^{(2)}\left[A \cup B \cup \{v\}\right]$ consists of a complete bipartite graph with partition sets A and B and a trivial connected component with vertex v. Hence all the edges in the edge-coloured subgraphs $\mathrm{g}\left[A \cup \{v\}\right]$ and $\mathrm{g}\left[B \cup \{v\}\right]$ are blue and therefore, by Proposition 5.2.15, $|A \cup \{v\}|, |B \cup \{v\}| \leq 3$. So $|V| = |X| + |A| + |B| + 1 \leq 8$. Moreover, if $|V| \geq 7$ then one of the sets A and B contains at least 2 elements. Assume without loss of generality that $|A| \geq 2$. Then $\mathrm{g}\left[X\right]$ and $\mathrm{g}\left[A \cup \{v\}\right]$ provides a pair of blue triangles with no common vertices. $\square$

Definition 5.2.31. For every index $i \in \{2,3,4,5\}$, define the edge-coloured graphs $\mathrm{g}_i = (\Gamma_i, c_i)$ such that $\Gamma_2 = \Gamma_3 := ([7], P_2([7]))$; $\Gamma_4 := ([8], P_2([8]))$; $\Gamma_5 := ([9], P_2([9]))$; c_i is an edge-colouring of Γ_i with 2 colours; and $\Gamma_i^{(1)}, \Gamma_i^{(2)}$ are the following graphs, respectively:

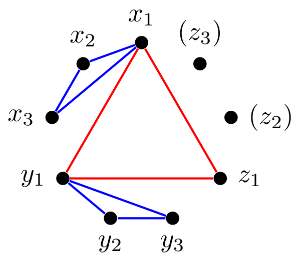


Proposition 5.2.32. *If $|V| \geq 7$, then g and g_i are isomorphic edge-coloured graphs for some index $i \in \{2,3,4,5\}$.*

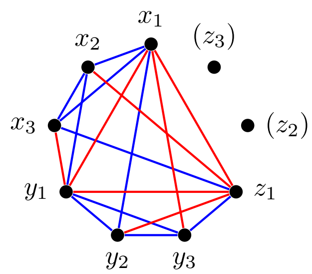
Proof. By Lemma 5.2.30, g contains a pair of monochromatic triangles of the same colour $i \in [2]$ with no common vertices. Assume without loss of generality that g contains two blue triangles with vertex sets $X = x_1x_2x_3$ and $Y = y_1y_2y_3$, respectively, with $X \cap Y = \varnothing$. Write $Z := V \setminus (X \cup Y) = z_1z_2z_3$ (z_1, z_2, z_3 are not necessarily distinct vertices) and get the following:



By Lemma 5.2.13, every red triangle in g contains one vertex of each of the sets X, Y and Z. Therefore we may assume that $\mathrm{g}\,[x_1y_1z_1]$ is a red triangle (because, for all permutations $\sigma \in S_X, \tau \in S_Y, \lambda \in S_Z$, $\sigma\tau\lambda$ is an automorphism of the edge-coloured graph above) and get the following:



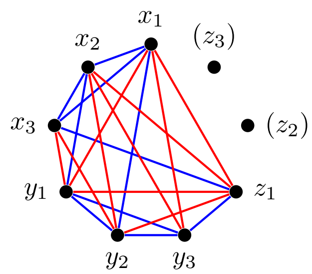
$\mathrm{g}\,[X \cup \{y_1, z_1\}]$, $\mathrm{g}\,[Y \cup \{x_1 z_1\}]$ and g_1 are isomorphic graphs. Therefore, there exist indices $i, j, p, q \in \{2, 3\}$ with $i \neq j$ and $p \neq q$ such that $x_iy_1, x_jz_1, y_px_1, y_qz_1$ are blue edges and $x_jy_1, x_iz_1, y_qx_1, y_pz_1$ are red edges. We may assume that $i = p = 2$ and $j = q = 3$ (because the permutation $(x_2x_3)(y_2y_3)$ is an automorphism of the edge-coloured graph above) and get the following:



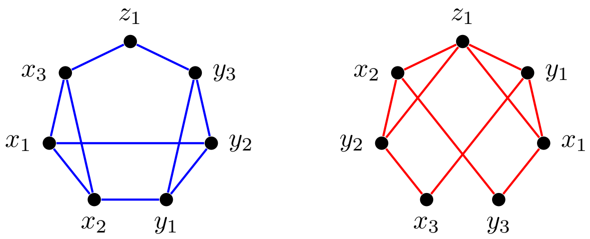
By Proposition 5.2.15, the following implications hold:

$$c(x_1y_2) = 1 \implies c(x_2y_2) = c(x_3y_2) = 2 \qquad \text{(so } \mathrm{g}\,[x_2y_2z_1] \text{ is a red triangle)}$$

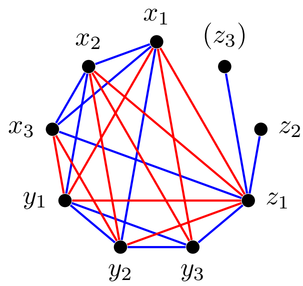
$$c(x_2y_1) = 1 \implies c(x_2y_3) = 2$$



If $|V| = 7$, we get:



According to the colour of the edge x_3y_3, if $|V| = 7$, then the edge-coloured graph above is isomorphic to g_2 or g_3. We may assume now that $|V| \geq 8$ and fix an index $i \in \{2, 3\}$. By Proposition 5.2.15, one of the edges z_ix_1, z_ix_2 is red. Observe that if $c(z_iz_1) = 2$ then, also by Proposition 5.2.15, $c(z_2x_1) = c(z_2x_2) = 1$, a contradiction. So we may conclude that $c(z_2z_1) = c(z_3z_1) = 1$ and get the following:

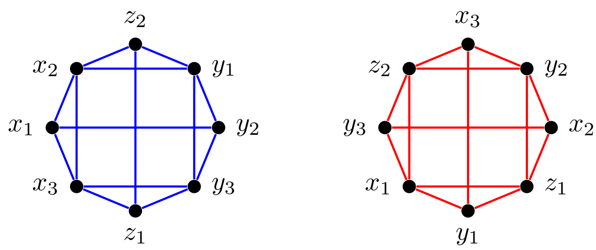


By Proposition 5.2.15, one of the edges z_2x_1, z_2x_2 is red. We may assume that $c(z_2x_1) = 2$ (because the permutation $(x_1x_2)(y_1y_2)$ is an automorphism of the edge-coloured graph above). By Proposition 5.2.15, the following implications hold:

$$c(z_2x_1) = 2 \implies c(z_2y_1) = 1 \implies c(z_2y_2) = c(z_2y_3) = 2 \implies c(z_2x_2) = 1 \implies c(z_2x_3) = 2$$

(so $\mathrm{g}[x_2y_1z_2]$ is a blue triangle;and $\mathrm{g}[x_1y_3z_2], \mathrm{g}[x_3y_2z_2]$ are red triangles)

Moreover, $c(x_3y_3) = 1$ because $\mathrm{g}[x_3y_2z_2]$ is a red triangle and $c(z_2y_3) = 2$. So $\mathrm{g}[x_3y_3z_1]$ is a blue triangle. If $|V| = 8$, we get:



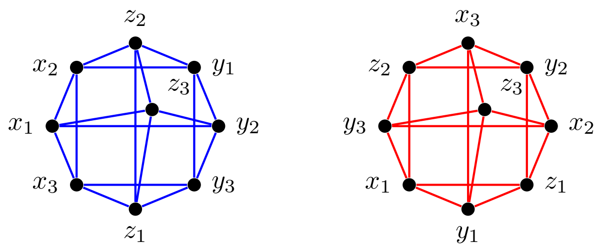
The edge-coloured graph above is isomorphic to g_4. We may assume now that $|V| = 9$. Recall that $c(z_3z_1) = 1$. By Proposition 5.2.15, the following implications hold:

$$c(z_3z_1) = 1 \implies c(z_3x_3) = c(z_3y_3) = 2$$

$$\implies c(z_3x_1) = c(z_3y_2) = c(z_3z_2) = 1 \implies c(z_3x_2) = c(z_3y_1) = 2$$

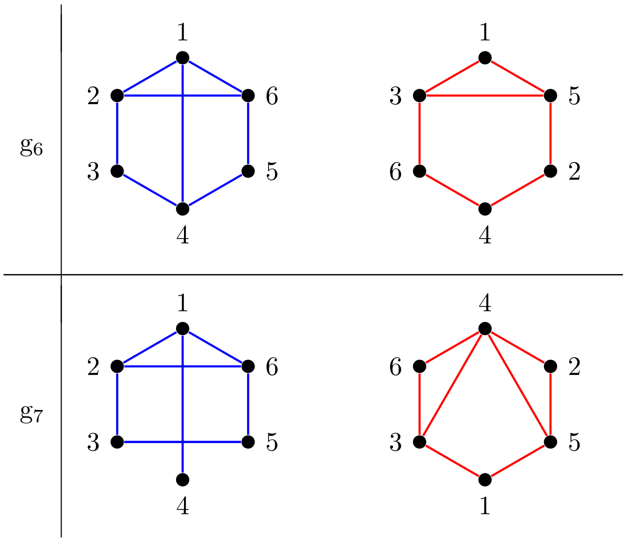
(so $\mathrm{g}[x_1y_2z_3], \mathrm{g}[z_1z_2z_3]$ are blue triangles; and $\mathrm{g}[x_2y_3z_3], \mathrm{g}[x_3y_1z_3]$ are red triangles)

Hence, we get:



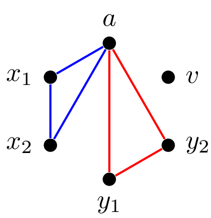
The edge-coloured graph above is isomorphic to g_5. $\square$

Definition 5.2.33. For every index $i \in \{6, 7\}$, define the edge-coloured graphs $\mathrm{g}_i = (\Gamma_i, c_i)$ such that $\Gamma_6 = \Gamma_7 = ([6], P_2[6])$, c_i is an edge-colouring of Γ_i with 2 colours; and $\Gamma_i^{(1)}, \Gamma_i^{(2)}$ are the following graphs, respectively:

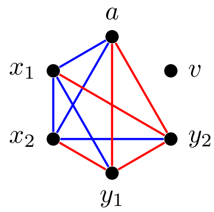


Proposition 5.2.34. *If $|V| = 6$, then g and g_i are isomorphic edge-coloured graphs, for some index $i \in \{6, 7\}$.*

Proof. By Remark 5.2.9 and Lemma 5.2.13, we may consider a blue and a red triangle in g with vertex sets $X = ax_1x_2$ and $Y = ay_2y_3$, respectively ($a, x_1, x_2, y_1, y_2 \in V$ are distinct vertices). Moreover, let v denote the single vertex in the set $V \setminus (X \cup Y)$. We get the following:

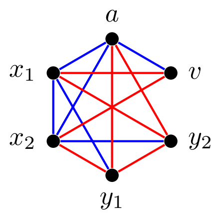


$\mathrm{g}\,[X \cup Y]$ and g_1 are isomorphic graphs. Therefore there exists an index $i \in [2]$ such that $x_iy_1, x_{3-i}y_2$ are blue edges and $x_{3-i}y_1, x_iy_2$ are red edges. We may assume that $i = 1$ (because the permutation (y_1y_2) is an automorphism of the edge-coloured graph above) and get the following:

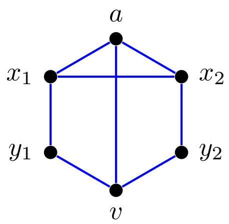


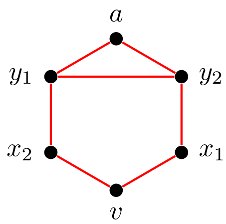
Now, we may assume that $c(va) = 1$ because the edge-coloured graph above and its colour inversion are isomorphic. By Proposition 5.2.15, the following implication holds:

$$c(va) = 1 \implies c(vx_1) = c(vx_2) = 2$$

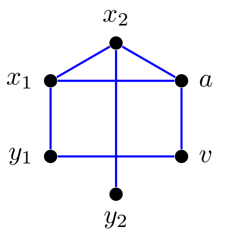


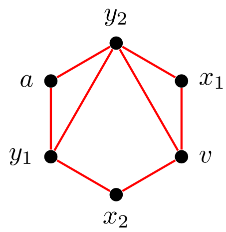
By Proposition 5.2.15, one of the edges vy_1, vy_2 is blue. Moreover, if exactly one of the edges vy_1, vy_2 is blue, we may assume that $c(vy_1) = 1$ and $c(vy_2) = 2$ (because $(x_1x_2)(y_1y_2)$ is an automorphism of the edge-coloured graph above). Now, if $c(vy_1) = c(vy_2) = 1$ we get:





On the other hand, if $c(vy_1) = 1$ and $c(vy_2) = 2$ we get:





The edge-coloured graphs above are isomorphic to g_6 and g_7, respectively. $\square$

Definition 5.2.35. Define the following simplicial complexes:

- $\mathcal{H}_1 := ([5], P_{\leq 3}([5]) \setminus \{125, 134\})$
- $\mathcal{H}_2 := ([7], P_{\leq 3}([7]) \setminus \{127, 135, 146, 234, 567\})$
- $\mathcal{H}_3 := ([7], P_{\leq 3}([7]) \setminus \{135, 146, 234, 567\})$
- $\mathcal{H}_4 := ([8], P_{\leq 3}([8]) \setminus \{128, 136, 147, 234, 257, 358, 456, 678\})$
- $\mathcal{H}_5 := ([9], P_{\leq 3}([9]) \setminus \{128, 136, 147, 159, 234, 257, 269, 358, 379, 456, 489, 678\})$
- $\mathcal{H}_6 := ([6], P_{\leq 3}([6]) \setminus \{126, 135\})$

- $\mathcal{H}_7 := ([6], P_{\leq 3}([6]) \setminus \{126, 245, 346\})$

Corollary 5.2.36. *Up to isomorphism, $\mathcal{H}$ is one of the simplicial complexes $\mathcal{H}_i$ for some index $i \in [7]$.*

Proof. By Corollary 5.2.23 and Propositions 5.2.26, 5.2.32 and 5.2.34, we get a characterization of the edge-coloured graph g as belonging to the finite set $\{\mathrm{g}_0, \mathrm{g}_1, \ldots, \mathrm{g}_7\}$ (up to isomorphism). Now we must find all simplicial complexes admitting the elements of the set $\mathrm{g}_0, \mathrm{g}_1, \ldots, \mathrm{g}_7$ as the edge-coloured graph satisfying Definition 5.2.4. In fact, we will show that there exists a unique simplicial complex $\mathcal{H}_i = (V_i, H_i)$ associated to each edge-coloured graph g_i, for every index $i \in \{0, \ldots, 7\}$; and that the simplicial complexes $\mathcal{H}_0$ and $\mathcal{H}_6$ constitute the only pair of isomorphic simplicial complexes.

Define, for every index $i \in \{0, \ldots, 7\}$, the sets $\mathcal{F}_i^{(1)}$ and $\mathcal{F}_i^{(2)}$ of all maximal cliques (with respect to set inclusion) of the graphs $\Gamma_i^{(1)}$ and $\Gamma_i^{(2)}$, respectively. Fix an index $i \in \{0, \ldots, 7\}$. We claim that, if g and g_i are isomorphic graphs, then $\mathcal{L}_M = \mathcal{F}_i^{(1)}$ and $\mathcal{L}_N = \mathcal{F}_i^{(2)}$ (we show only the first equality because the arguments are similar). To prove the claim, consider a line $L_0 \in \mathcal{L}_M$ and a maximal clique C_0 of $\Gamma_i^{(1)}$ containing L_0. Note that $C_0 \neq V$ because otherwise $\mathcal{L}_N = \varnothing$ and so, by Remark 4.1.4, $\mathcal{H} \cong \mathcal{U}_{2,n}$: a contradiction because $\mathcal{L}_M \neq \varnothing$. Suppose that there exists a vertex $c \in C_0 \setminus L_0$. By Proposition 4.1.2, for every line $L \in \mathcal{L}_M \setminus \{L_0\}$ such that $c \in L$, $|L \cap L_0| \leq 1$. Therefore $\ell c \in \mathcal{L}_M$ for every element $\ell \in L_0$. But then, by Proposition 4.1.6, for all distinct vertices $x, y \in L_0$, we have $xyc \in H_M$ (because $|xyc \cap xc| = 2$ and $xc \in \mathcal{L}_M$) and $xyc \notin H_N$ (because xyc is a 3-anticlique of $\Gamma_i^{(2)} = \mathscr{G}(N)$): a contradiction because $H_M = H_N$. So we may conclude that $L_0 = C_0$ is a maximal clique of $\Gamma_i^{(1)}$ and therefore $\mathcal{L}_M \subseteq \mathcal{F}_i^{(1)}$.

Conversely, consider a maximal clique $C_0 \in \mathcal{F}_i^{(1)}$ and observe that $|C_0| \in \{2, 3\}$. If $|C_0| = 2$ then it is clear that $C_0 \in \mathcal{L}_M$. We may then assume that $|C_0| = 3$. As $\mathrm{g}[C_0]$ is a monochromatic triangle, then, by Lemma 5.2.10, $C_0 \notin H = H_M$. Then, by Proposition 4.1.6, $C_0 \in P_3(L)$ for some line $L \in \mathcal{L}_M$. But then $C_0 = L$ by maximality of C_0 and so $C_0 \in \mathcal{L}_M$. We may conclude that $\mathcal{F}_i^{(1)} \subseteq \mathcal{L}_M$ and the claim holds. Therefore,

$$\bigcup_{L \in \mathcal{L}_M} P_3(L) = \bigcup_{C \in \mathcal{F}_i^{(1)}} P_3(C) = \left\{\text{triangles of } \Gamma_i^{(1)}\right\} = \{\text{blue triangles of } \mathrm{g}_i\}$$

$$\bigcup_{J \in \mathcal{L}_N} P_3(J) = \bigcup_{D \in \mathcal{F}_i^{(2)}} P_3(D) = \left\{\text{triangles of } \Gamma_i^{(2)}\right\} = \{\text{red triangles of } \mathrm{g}_i\}$$

So, by Proposition 4.1.6,

$$\begin{aligned} H_i &= P_{\leq 3}(V_i) \setminus \left(\left\{\text{3-anticliques of } \Gamma_i^{(1)}\right\} \cup \{\text{blue triangles of } \mathrm{g}_i\}\right) \\ &= P_{\leq 3}(V_i) \setminus (\{\text{red triangles of } \mathrm{g}_i\} \cup \{\text{blue triangles of } \mathrm{g}_i\}) \qquad \text{(if } \Gamma \text{ is complete, i.e., if } i \neq 0) \end{aligned}$$

Observe that $V_0 = V_6 = [6]$ and $H_0 = H_6 = \{126, 135\}$ so the simplicial complexes $\mathcal{H}_0$ and $\mathcal{H}_6$ are isomorphic. Moreover, for every index $i \in [7]$, the simplicial complex $\mathcal{H}_i$ constructed in this proof coincides with the simplicial complex $\mathcal{H}_i$ in Definition 5.2.35. Observe that there are no other pairs of isomorphic simplicial complexes because the number of points or simplexes is distinct. □

Corollary 5.2.37. *$\mathcal{H}$ is a matroid.*

Proof. Observe that, by Proposition 3.3.6, $\mathcal{L}_M, \mathcal{L}_N \subseteq \mathcal{L}_{\mathrm{Mat}\,\mathcal{H}}$. Therefore the edges of the graphs $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are also edges of the graph $\mathscr{G}(\mathrm{Mat}\,\mathcal{H})$. Therefore, $\mathscr{G}(\mathrm{Mat}\,\mathcal{H})$ is a complete graph. Hence, by Proposition 4.1.9, $\mathcal{H}$ is a matroid. □

Proposition 5.2.38. *The simplicial complexes $\mathcal{H}_1, \ldots, \mathcal{H}_7$ admit alternative 1-complete boolean representations with disjoint sets of lines.*

Proof. We construct boolean representations for the simplicial complex $\mathcal{H}_1$, the remaining cases being analogous. Following the arguments of the proof of Corollary 5.2.36, we may construct 1-complete boolean matrices matrices having as set of lines precisely the set of maximal cliques of graphs $\Gamma_1^{(1)}$ and $\Gamma_1^{(2)}$. We obtain the following matrices satisfying the required properties:

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

□

Chapter 6

Distance to the Uniform Matroid $\mathcal{U}_{2,n}$: upper bounds

6.1 General Results

Context 6.1.1. We start this Chapter by presenting some needed technical definitions and results relating boolean representations and graphs, so we may use arguments from graph theory to get upper bounds for distances between simplicial complexes.

Notation 6.1.2. Given a graph $\Gamma = (V, E)$, use the notation $e(\Gamma) := |E|$.

Definition 6.1.3. Given a graph $\Gamma = (V, E)$, the *matrix associated to the graph* Γ, $\mathscr{M}(\Gamma)$ is the 1-complete (short) boolean matrix with columns indexed by V and such that $\mathcal{L}_{\mathscr{M}(\Gamma)} = E$.

Definition 6.1.4. Given a simplicial complex $\mathcal{H} = (V, H) \in \mathrm{BPav}_2$, define $\mathcal{F}_{\mathcal{H}} := P_{\leq 3}(V) \setminus H$ and the set $\mathcal{A}_{\mathcal{H}}$ of all graphs with vertex set V having as triangles exactly the elements of $\mathcal{F}_{\mathcal{H}}$. Write $\mathcal{A}^c_{\mathcal{H}} := \{\Gamma^c \mid \Gamma \in \mathcal{A}_{\mathcal{H}}\}$.

Notation 6.1.5. Given a simplicial complex $\mathcal{H}$, use the notation

$$\mathrm{BR}'(\mathcal{H}) := \mathrm{BR}_{\text{1-complete}}(\mathcal{H}) \cap \mathrm{BR}_{\text{short}}(\mathcal{H}).$$

Proposition 6.1.6. *Given a 1-complete short matrix M and a graph Γ, the following properties hold:*

1. $\mathscr{M}(\mathscr{G}(M))$ *and* M *are congruent matrices*
2. $\mathscr{G}(\mathscr{M}(\Gamma)) = \Gamma$
3. $e(\mathscr{G}(M)) = |\mathcal{L}_M|$
4. $\left|\mathcal{L}_{\mathscr{M}(\Gamma)}\right| = e(\Gamma)$

Proof. Follows clearly from definitions. □

Proposition 6.1.7. *Given a simplicial complex $\mathcal{H} = (V, H) \in \mathrm{BPav}_2$, the maps*

$$\begin{array}{rcl} \mathrm{BR}'(\mathcal{H})/\cong & \to & \mathcal{A}^c_{\mathcal{H}} \\ M & \mapsto & \mathscr{G}(M) \end{array} \qquad\qquad \begin{array}{rcl} \mathcal{A}^c_{\mathcal{H}} & \to & \mathrm{BR}'(\mathcal{H})/\cong \\ \Gamma & \mapsto & \mathscr{M}(\Gamma) \end{array}$$

are inverse bijections.

Proof. Observe that $\mathcal{A}^c_{\mathcal{H}}$ is the set of all graphs with vertex set V having as 3-anticliques exactly the elements of $\mathcal{F}_{\mathcal{H}}$. Consider a matrix $M \in \mathrm{BR}'(\mathcal{H})$ and a graph $\Gamma \in \mathcal{A}^c_{\mathcal{H}}$. By Proposition 6.1.6, it suffices to show that $\mathscr{G}(M) \in \mathcal{A}^c_{\mathcal{H}}$ and $\mathscr{M}(\Gamma) \in \mathrm{BR}'(\mathcal{H})$. By Proposition 4.1.6:

$$\begin{aligned} P_{\leq 3}(V) \setminus \mathcal{F}_{\mathcal{H}} = H = H_M & \\ &= P_{\leq 3}(V) \setminus \left(\{\text{3-anticliques of } \mathscr{G}(M)\} \cup \bigcup_{L \in \mathcal{L}_M} P_3(L) \right) \\ &\qquad\qquad \text{(because } M \in \mathrm{BR}(\mathcal{H}) \text{ and } \mathcal{H} \in \mathrm{BPav}_2) \\ &= P_{\leq 3}(V) \setminus \{\text{3-anticliques of } \mathscr{G}(M)\} \qquad \text{(because } M \text{ is short)} \end{aligned}$$

Hence $\{\text{3-anticliques of } \mathscr{G}(M)\} = \mathcal{F}_{\mathcal{H}}$, i.e. $\mathscr{G}(M) \in \mathcal{A}^c_{\mathcal{H}}$. Moreover, observe that $\mathcal{L}_{\mathscr{M}(\Gamma)} = E$ is a PEG and therefore, by Proposition 4.1.5, $\mathcal{H}_{\mathscr{M}(\Gamma)} \in \mathrm{BPav}_2$. Then, by Proposition 4.1.6, we also have:

$$\begin{aligned} H_{\mathscr{M}(\Gamma)} &= P_{\leq 3}(V) \setminus \left(\{\text{3-anticliques of } \mathscr{G}(\mathscr{M}(\Gamma))\} \cup \bigcup_{L \in \mathcal{L}_{\mathscr{M}(\Gamma)}} P_3(L) \right) \quad \text{(because } \mathcal{H}_{\mathscr{M}(\Gamma)} \in \mathrm{BPav}_2) \\ &= P_{\leq 3}(V) \setminus \{\text{3-anticliques of } \Gamma\} \qquad \text{(because } M \text{ is short and } \mathscr{G}(\mathscr{M}(\Gamma)) = \Gamma) \\ &= P_{\leq 3}(V) \setminus \mathcal{F}_{\mathcal{H}} \qquad \text{(because } \Gamma \in \mathcal{A}^c_{\mathcal{H}}) \end{aligned}$$

Hence $H_{\mathscr{M}(\Gamma)} = H$, i.e., $\mathscr{M}(\Gamma) \in \mathrm{BR}'(\mathcal{H})$. □

Lemma 6.1.8. *Given $n \in \mathbb{N}$ with $n \geq 3$ and a simplicial complex $\mathcal{H} = (V, H) \in \mathrm{BPav}_2^{(n)}$,*

$$d_n(\mathcal{U}_{2,n}, \mathcal{H}) \leq \binom{n}{2} - \max_{\Gamma \in \mathcal{A}_{\mathcal{H}}} e(\Gamma).$$

Proof.

$$\begin{aligned} d_n(\mathcal{U}_{2,n}, \mathcal{H}) &\leq \min_{M \in \mathrm{BR}'(\mathcal{H})} |\mathcal{L}_M| \\ &= \min_{\Gamma \in \mathcal{A}^c_{\mathcal{H}}} e(\Gamma) \qquad \text{(by Propositions 6.1.6 and 6.1.7)} \\ &= \min_{\Gamma \in \mathcal{A}_{\mathcal{H}}} e(\Gamma^c) = \binom{n}{2} - \max_{\Gamma \in \mathcal{A}_{\mathcal{H}}} e(\Gamma) \end{aligned}$$

□

Remark 6.1.9. Given $n \in \mathbb{N}$, the following equalities hold:

- $\left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor \right) = \left\lfloor \frac{n^2}{4} \right\rfloor$
- $\binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$
- $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor$

- $\left\lfloor \dfrac{\lfloor n/2 \rfloor}{2} \right\rfloor = \left\lfloor \dfrac{n}{4} \right\rfloor$
- $\left\lfloor \dfrac{n}{2} \right\rfloor - \left\lfloor \dfrac{n}{4} \right\rfloor = \left\lfloor \dfrac{n+2}{4} \right\rfloor$
- $n - \left\lfloor \dfrac{n}{2} \right\rfloor = \left\lfloor \dfrac{n+1}{2} \right\rfloor$

Proof.

$$\left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor\right) = \begin{cases} \ell^2 & \text{if } n = 2\ell \text{ for some } \ell \in \mathbb{N} \\ \ell(\ell+1) = \ell^2 + \ell & \text{if } n = 2\ell+1 \text{ for some } \ell \in \mathbb{N} \end{cases} = \left\lfloor \frac{n^2}{4} \right\rfloor$$

$$\binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor = \begin{cases} (2\ell^2 - \ell) - \ell^2 = \ell(\ell-1) & \text{if } n = 2\ell \text{ for some } \ell \in \mathbb{N} \\ (2\ell^2 + \ell) - (\ell^2 + \ell) = \ell^2 & \text{if } n = 2\ell+1 \text{ for some } \ell \in \mathbb{N} \end{cases} = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$$

$$\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} (\ell^2 - \ell) + \ell = \ell^2 & \text{if } n = 2\ell \text{ for some } \ell \in \mathbb{N} \\ \ell^2 + \ell & \text{if } n = 2\ell+1 \text{ for some } \ell \in \mathbb{N} \end{cases} = \left\lfloor \frac{n^2}{4} \right\rfloor$$

$$\left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor = \ell \text{ if } n \in \{4\ell, 4\ell+1, 4\ell+2, 4\ell+3\} \text{ for some } \ell \in \mathbb{N} = \left\lfloor \frac{n}{4} \right\rfloor$$

$$\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor = \begin{cases} \ell & \text{if } n \in \{4\ell, 4\ell+1\} \text{ for some } \ell \in \mathbb{N} \\ \ell+1 & \text{if } n \in \{4\ell+2, 4\ell+3\} \text{ for some } \ell \in \mathbb{N} \end{cases} = \left\lfloor \frac{n+2}{4} \right\rfloor$$

$$n - \left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \ell & \text{if } n = 2\ell \text{ for some } \ell \in \mathbb{N} \\ \ell+1 & \text{if } n = 2\ell+1 \text{ for some } \ell \in \mathbb{N} \end{cases} = \left\lfloor \frac{n+1}{2} \right\rfloor$$

□

6.2 $d_n(\mathcal{U}_{2,n}, \mathcal{U}_{3,n})$

Context 6.2.1. The concept of minimum degree (mindeg) provides a first approach to get upper bounds to the distance $d_n(\mathcal{U}_{2,n}, \mathcal{U}_{3,n})$, so we compute $\mathrm{mindeg}\,\mathcal{U}_{3,n}$ for $n \in \{3,4,5\}$. Then we use Mantel's Theorem to get a quadratic upper bound. Finally, in Proposition 6.2.10 we get an improvement (still quadratic) of this upper bound by using results from Chapter 5.

Definition 6.2.2. Given a boolean representable simplicial complex $\mathcal{H}$, the *minimum degree* of $\mathcal{H}$, $\mathrm{mindeg}\,\mathcal{H}$ is the minimum number of rows of a boolean representation of $\mathcal{H}$.

Lemma 6.2.3. *Given a boolean representable simplicial complex $\mathcal{H}$, $\mathrm{mindeg}\,\mathcal{H} \geq \dim \mathcal{H} + 1$.*

Proof. Write $\mathcal{H} = (V, H)$; consider a simplex $X \in P_{\dim \mathcal{H}+1}(V) \cap H$ and a matrix $M \in \mathrm{BR}(\mathcal{H})$ with rows indexed by a set R. Then, up to congruence, there exists a subset $R' \subseteq R$ such that the submatrix $M[R', X]$ is lower unitriangular. Hence $|R| \geq |R'| = |X| = \dim \mathcal{H} + 1$. □

Proposition 6.2.4. $\mathrm{mindeg}\,\mathcal{U}_{3,3} = \mathrm{mindeg}\,\mathcal{U}_{3,4} = 3$ *and* $\mathrm{mindeg}\,\mathcal{U}_{3,5} = 5$.

Proof. By Lemma 6.2.3, $\mathrm{mindeg}\,\mathcal{U}_{3,3}, \mathrm{mindeg}\,\mathcal{U}_{3,4}, \mathrm{mindeg}\,\mathcal{U}_{3,5} \geq 3$. Define matrices (with respective row and column indexing):

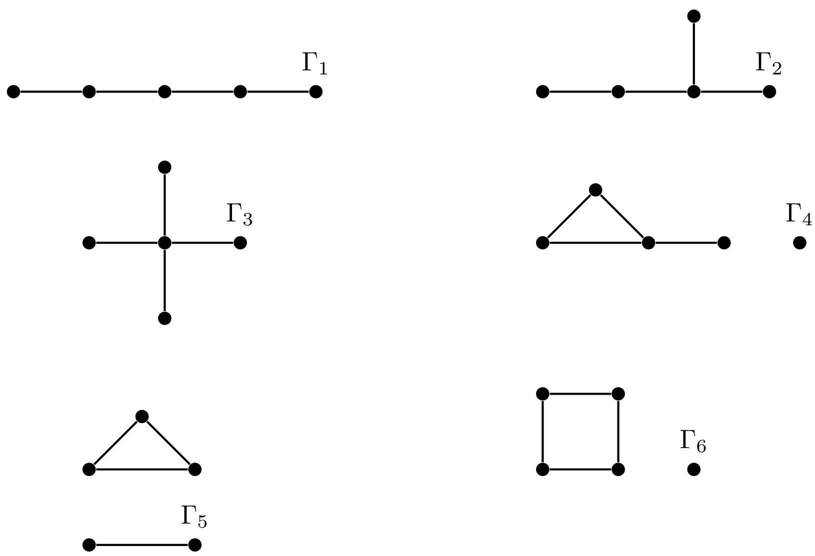
$$M := \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array} \qquad N := \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad Q := \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

M is lower unitriangular so $123 \in H_M$. Hence $H_M = P_{\leq 3}([3])$. So $M \in \mathrm{BR}(\mathcal{U}_{3,3})$ and therefore $\mathrm{mindeg}\,\mathcal{U}_{3,3} \leq 3$ and equality holds.

Observe that the maximum number of entries equal to 0 in each row of the matrices N and Q is 2 so $|X| \leq 3$ for every set $X \in H_N \cup H_Q$. Now, the following submatrices are lower unitriangular (if we fix the order of rows and columns): $N[s_1s_2s_3, 123]$, $N[s_3s_2s_1, 124]$, $N[s_2s_3s_1, 134]$ and $N[s_1s_3s_2, 432]$; so $P_3([4]) \subseteq H_N$ and therefore $H_N = P_{\leq 3}([4])$. Hence $N \in \mathrm{BR}(\mathcal{U}_{3,4})$, we have $\mathrm{mindeg}\,\mathcal{U}_{3,4} \leq 3$ and equality holds.

The following submatrices are lower unitriangular (if we fix the order of rows and columns): $Q[t_2t_3t_4, 123]$, $Q[t_1t_5t_4, 421]$, $Q[t_1t_5t_4, 521]$, $Q[t_3t_4t_5, 134]$, $Q[t_5t_4t_3, 315]$, $Q[t_4t_5t_1, 145]$, $Q[t_3t_4t_5, 234]$, $Q[t_2t_1t_3, 532]$, $Q[t_4t_5t_1, 245]$, $Q[t_4t_5t_1, 345]$; so $P_3([5]) \subseteq H_Q$ and therefore $H_Q = P_{\leq 3}([5])$. Hence $Q \in \mathrm{BR}(\mathcal{U}_{3,5})$ and we have $\mathrm{mindeg}\,\mathcal{U}_{3,5} \leq 5$.

Suppose now that there exists a matrix $W \in \mathrm{BR}(\mathcal{U}_{3,5})$ with at most 4 rows indexed by a set R. For every $r \in R$, $Z_r^W \in \mathrm{Fl}\,\mathcal{U}_{3,5}$ by Lemma 3.3.5; so $\left|Z_r^W\right| \leq 2$ by Lemma 3.5.6; hence $e(\mathscr{G}(W)) \leq 4$. So $e(\mathscr{G}(W))$ is a subgraph of one of the following graphs (up to isomorphism):



Observe now that $\mathcal{U}_{3,5} \in \mathrm{BPav}_2$ and $W \in \mathrm{BR}(\mathcal{U}_{3,5})$. So, by Proposition 4.1.6

$$P_{\leq 3}([5]) \setminus \{\text{3-anticliques of } \mathscr{G}(W)\} = P_{\leq 3}([5])$$

and therefore there are no 3-anticliques in $\mathscr{G}(W)$. Hence $\mathscr{G}(M)$ is isomorphic to Γ_5 and so, up to congruence, M is the matrix

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

But then $45 \notin \mathcal{H}_M$, a contradiction. So $\operatorname{mindeg}\mathcal{U}_{3,5} \geq 5$ and equality holds. □

Corollary 6.2.5. *$d_n(\mathcal{U}_{3,3}, \mathcal{U}_{2,3}), d_n(\mathcal{U}_{3,4}, \mathcal{U}_{2,4}) \leq 3$ and $d_n(\mathcal{U}_{3,5}, \mathcal{U}_{2,5}) \leq 5$.* □

Theorem 6.2.6 (Mantel, 1907 [4]). *Given $n \in \mathbb{N}$,*

$$\max_{\Gamma \in \mathcal{A}_{\mathcal{U}_{3,n}}} e(\Gamma) = \left\lfloor \frac{n^2}{4} \right\rfloor$$

Proof. Write $\mathcal{A}_n := \mathcal{A}_{\mathcal{U}_{3,n}}$ and $\mathrm{m}_n := \max_{\Gamma \in \mathcal{A}_n} e(\Gamma)$. Proceed by strong induction on n, the case $n = 1$ being trivial. Suppose that $n \geq 2$ and the result holds for every $m < n$. Fix some element $a \in \mathbb{N}$ with $a < n$ and consider the set $\mathcal{B}_{n,a} \subseteq \mathcal{A}_n$ whose elements are bipartite graphs with a partition set of size a. Note that, for every $\Gamma \in \mathcal{B}_{n,a}$, $e(\Gamma) \leq e(K_{a,n-a})$. Consider now the set $\mathcal{B}_n := \bigcup_{a<n} \mathcal{B}_{n,a} \subseteq \mathcal{A}_n$ whose elements are bipartite and observe that, by the previous argument,

$$\mathrm{m}'_n := \max_{\Gamma \in \mathcal{B}_n} e(\Gamma) = \max_{a<n} e(K_{a,n-a}) = \max_{a<n} a(n-a)$$

If n is even, m'_n is attained by taking $a = n/2$; if n is odd, m'_n is attained by taking $a = (n-1)/2$ or $a = (n+1)/2$. So, m'_n is attained by taking $\lfloor n/2 \rfloor$ and we get, applying Remark 6.1.9:

$$\mathrm{m}'_n = \left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor\right) = \left\lfloor \frac{n^2}{4} \right\rfloor$$

Consider now a graph $\Gamma \in \mathcal{A}_n \setminus \mathcal{B}_n$ which is triangle-free but not bipartite. Then there exists a cycle of minimal odd length $k \geq 5$ in Γ and we may consider the subgraph C induced by its vertices and the subgraph R induced by the remaining vertices. Observe that $e(C) = k$ by minimality of k and $e(R) \leq \lfloor (n-k)^2/4 \rfloor \leq (n-k)^2/4$ by induction hypothesis. Moreover, for every vertex $r \in R$, there exist at most $(k-1)/2$ edges joining r and C because Γ is triangle-free. Hence, there exist at most $(n-k)(k-1)/2$ edges joining R and C. So,

$$\begin{aligned} e(\Gamma) &\leq k + \frac{(n-k)^2}{4} + \frac{(n-k)(k-1)}{2} = \frac{n^2 - 2n - k^2 + 6k}{4} \\ &\leq \frac{n^2 - 2k - k^2 + 6k}{4} && \text{(because } k \leq n\text{)} \\ &= \frac{n^2 - k(k-4)}{4} \leq \frac{n^2 - 1}{4} && \text{(because } k \geq 5\text{)} \\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor \end{aligned}$$

□

Corollary 6.2.7. *Given $n \in \mathbb{N}$*

$$d_n(\mathcal{U}_{2,n}, \mathcal{U}_{3,n}) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$$

Proof.

$$\begin{aligned}
d_n(\mathcal{U}_{2,n},\mathcal{U}_{3,n}) &\leq \binom{n}{2} - \max_{\Gamma\in\mathcal{A}_{\mathcal{U}_{3,n}}} e(\Gamma) && \text{(by Lemma 6.1.8)}\\
&= \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor && \text{(by Theorem 6.2.6)}\\
&= \left\lfloor \frac{(n-1)^2}{4} \right\rfloor && \text{(by Remark 6.1.9)}
\end{aligned}$$

□

Notation 6.2.8. Given a set V and disjoint subsets $A, B \subseteq V$, use the notation $AB := \{ab \mid a \in A, b \in B\} \subseteq P_2(A \cup B)$ and note that $\Gamma = (A \cup B, AB)$ is a complete bipartite graph with partition sets A and B.

Lemma 6.2.9. *Given $n \in \mathbb{N}$ with $n \geq 3$, write $k := \lfloor n/2 \rfloor$, $\ell := \lfloor k/2 \rfloor$, $p := n - \lfloor n/2 \rfloor$ and $q := \lfloor p/2 \rfloor$. Define sets*

- $V_1 := [\ell]$
- $V_2 := [k] \setminus [\ell]$
- $V_3 := [k+q] \setminus [k]$
- $V_4 := [n] \setminus [k+q]$
- $E_1 := V_1V_3 \setminus \{\{i, k+i\} \mid i \in [\ell]\}$
- $E_2 := V_2V_4 \setminus \{\{\ell+i, k+q+i\} \mid i \in [k-\ell]\}$
- $E := P_2([k]) \cup P_2([n] \setminus [k]) \cup E_1 \cup E_2$
- $E' := V_1V_2 \cup V_3V_4 \cup E_1 \cup E_2$

Define graphs $\Gamma := ([n], E)$, $\Gamma' := ([n], E')$ and matrices $M := \mathscr{M}(\Gamma)$, $M' := \mathscr{M}(\Gamma')$. Then

$$d_n(\mathcal{H}_M, \mathcal{H}_{M'}) \leq \binom{\lfloor n/4 \rfloor}{2} + \binom{\lfloor (n+1)/4 \rfloor}{2} + \binom{\lfloor (n+2)/4 \rfloor}{2} + \binom{\lfloor (n+3)/4 \rfloor}{2}$$

and $\mathcal{H}_M \cong \mathcal{U}_{3,n}$.

Proof. Define the set

$$F := P_2(V_1) \cup P_2(V_2) \cup P_2(V_3) \cup P_2(V_4)$$

and observe that $E' = E \setminus F$. Therefore the matrix M' can be obtained from the matrix M by removing the lines of M corresponding to edges on F (using the bijection established in Proposition 6.1.7). Hence, if we sequentially remove the lines of M corresponding to edges of F, we get a path on the graph $\mathcal{D}_n$ with length at most $|F|$. So the following equalities hold:

$$\begin{aligned}
d_n(\mathcal{H}_M, \mathcal{H}_{M'}) &\leq |F|\\
&= \binom{|V_1|}{2} + \binom{|V_2|}{2} + \binom{|V_3|}{2} + \binom{|V_4|}{2}\\
&= \binom{\ell}{2} + \binom{k-\ell}{2} + \binom{q}{2} + \binom{p-q}{2}
\end{aligned}$$

$$= \binom{\lfloor n/4 \rfloor}{2} + \binom{\lfloor (n+2)/4 \rfloor}{2} + \binom{\lfloor (n+1)/4 \rfloor}{2} + \binom{\lfloor (n+3)/4 \rfloor}{2}$$

(by Remark 6.1.9)

Moreover, observe that there are no 3-anticliques in Γ so

$$H_M = P_{\le 3}([n]) \setminus \{\text{3-anticliques of } \Gamma\} = P_{\le 3}([n])$$

Hence $\mathcal{H}_M \cong \mathcal{U}_{3,n}$. □

Proposition 6.2.10. *Given $n \in \mathbb{N}$ with $n \ge 3$,*

$$\begin{aligned} d_n(\mathcal{U}_{2,n},\mathcal{U}_{3,n}) &\le \binom{\lfloor n/4 \rfloor}{2} + \binom{\lfloor (n+1)/4 \rfloor}{2} + \binom{\lfloor (n+2)/4 \rfloor}{2} + \binom{\lfloor (n+3)/4 \rfloor}{2} + 2 \\ &= 4\binom{\lfloor n/4 \rfloor}{2} + \left(n - 4\left\lfloor \frac{n}{4} \right\rfloor\right)\left\lfloor \frac{n}{4} \right\rfloor + 2 \end{aligned}$$

Proof. Use all the notation introduced in Lemma 6.2.9 and define the set of edges

$$G := \{\{i, k+i\} \mid i \in [\ell]\} \cup \{\{\ell+i, k+q+i\} \mid i \in [k-\ell]\}$$

We claim that G is a matching with respect to the complete bipartite graph with partition sets $V_1 \cup V_4$ and $V_2 \cup V_3$. Note that the edges in G are clearly pairwise disjoint. It remains to show that G is well defined, i.e. the following properties hold:

1. $k+i \in V_3$ for every $i \in [\ell]$
2. $k+q+i \in V_4$ for every $i \in [k-\ell]$

Now, observe that condition 1 holds if and only if $|V_3| \ge |V_1|$ and condition 2 holds if and only if $|V_4| \ge |V_2|$. Using Remark 6.1.9 we may conclude that

$$|V_3| = q = \left\lfloor \frac{n+1}{4} \right\rfloor \ge \left\lfloor \frac{n}{4} \right\rfloor = \ell = |V_1|$$
$$|V_4| = p - q = \left\lfloor \frac{n+3}{4} \right\rfloor \ge \left\lfloor \frac{n+2}{4} \right\rfloor = k - \ell = |V_2|$$

and therefore the claim holds. Now, observe that Γ' is obtained from the complete bipartite graph with partition sets $V_1 \cup V_4$ and $V_2 \cup V_3$ by removing the matching G. Hence, by Proposition 5.1.17, the simplicial complex $\mathcal{H}_{M'}$ admits a 1-complete boolean representation with two lines and therefore $d_n(\mathcal{U}_{2,n}, \mathcal{H}_{M'}) \le 2$. So the following equalities hold:

$$\begin{aligned} d_n(\mathcal{U}_{2,n},\mathcal{U}_{3,n}) &\le d_n(\mathcal{U}_{2,n},\mathcal{H}_{M'}) + d_n(\mathcal{H}_{M'},\mathcal{U}_{3,n}) \\ &\le 2 + d_n(\mathcal{H}_{M'},\mathcal{U}_{3,n}) && \text{(by the previous computation)} \\ &\le 2 + \binom{\lfloor n/4 \rfloor}{2} + \binom{\lfloor (n+1)/4 \rfloor}{2} + \binom{\lfloor (n+2)/4 \rfloor}{2} + \binom{\lfloor (n+3)/4 \rfloor}{2} \end{aligned}$$

(by Lemma 6.2.9)

Moreover, if we write $n = 4\ell + r$ for some integers ℓ, r with $0 \le r < 4$, then the following equalities hold:

$$
\begin{aligned}
&2+\binom{\lfloor n/4\rfloor}{2}+\binom{\lfloor (n+1)/4\rfloor}{2}+\binom{\lfloor (n+2)/4\rfloor}{2}+\binom{\lfloor (n+3)/4\rfloor}{2}\\
&=\begin{cases}
2+4\binom{\ell}{2} & \text{if } r=0\\
2+3\binom{\ell}{2}+\binom{\ell+1}{2}=2+4\binom{\ell}{2}+\binom{\ell}{1}=2+4\binom{\ell}{2}+\ell & \text{if } r=1\\
2+2\binom{\ell}{2}+2\binom{\ell+1}{2}=2+4\binom{\ell}{2}+2\binom{\ell}{1}=2+4\binom{\ell}{2}+2\ell & \text{if } r=2\\
2+\binom{\ell}{2}+3\binom{\ell+1}{2}=2+4\binom{\ell}{2}+3\binom{\ell}{1}=2+4\binom{\ell}{2}+3\ell & \text{if } r=3
\end{cases}\\
&=2+4\binom{\lfloor n/4\rfloor}{2}+\left(n-4\left\lfloor\frac{n}{4}\right\rfloor\right)\left\lfloor\frac{n}{4}\right\rfloor
\end{aligned}
$$

□

6.3 $d\left(\mathcal{U}_{2,n},\mathcal{U}_{3,n}\right)$

Context 6.3.1. Throughout this Section, we intend to get upper bounds to the distance $d(\mathcal{U}_{2,n},\mathcal{U}_{3,n})$.

Lemma 6.3.2. *Given $n\in\mathbb{N}$ with $n\geq 3$, consider the 1-complete (short) boolean matrices M and N with columns indexed by $[n]$ such that $\mathcal{L}_M=P_2([n])$ and $\mathcal{L}_N=P_2([n-1])$. Then $M,N\in\mathrm{BR}(\mathcal{U}_{3,n})$.*

Proof. Observe that $\mathcal{L}_M,\mathcal{L}_N$ are PEGs so $\mathcal{H}_M,\mathcal{H}_N\in\mathrm{BPav}_2$ by Proposition 4.1.5 and therefore we may apply Proposition 4.1.6. Now, up to isomorphism, $\mathscr{G}(M)$ is the complete graph with n vertices and $\mathscr{G}(N)$ can be obtained from the complete graph with $n-1$ vertices by adding a trivial connected component. Therefore there are no 3-anticliques in $\mathscr{G}(M)$ and $\mathscr{G}(N)$ and so, by Proposition 4.1.6, $H_M=H_N=P_{\leq 3}([n])$. Hence $M\in\mathrm{BR}(\mathcal{U}_{3,n})$. □

Proposition 6.3.3. *Given $n\in\mathbb{N}$ with $n\geq 3$, $d(\mathcal{U}_{3,n},\mathcal{U}_{2,n})\leq 2n-5$.*

Proof. Write $I(n):=2n-5$ and proceed by induction on n. Suppose that $n=3$. Define 1-complete boolean matrices M and N with columns indexed by $[3]$ and such that $\mathcal{L}_M=\{12\}$ and $\mathcal{L}_N=\varnothing$. Then $N\in\mathrm{BR}(\mathcal{U}_{2,3})$ and, by Lemma 6.3.2, $M\in\mathcal{U}_{3,3}$. Observe that $\mathcal{U}_{2,3}$ and $\mathcal{U}_{3,3}$ are distinct simplicial complexes and $|\mathcal{L}_M\triangle\mathcal{L}_N|=1$ so $d(\mathcal{U}_{3,3},\mathcal{U}_{2,3})=d_3(\mathcal{U}_{3,3},\mathcal{U}_{2,3})=1=I(3)$.

Consider now $n\geq 4$ and suppose that $d(\mathcal{U}_{3,n-1},\mathcal{U}_{2,n-1})\leq I(n-1)$. Define the following 1-complete boolean matrices:

- M_1 with columns indexed by $[n]$ and such that $\mathcal{L}_{M_1}=P_2([n-1])$
- M_2 with columns indexed by $[n-1]$ and such that $\mathcal{L}_{M_2}=P_2([n-1])$
- N_1 with columns indexed by $[n-1]$ and such that $\mathcal{L}_{N_1}=\varnothing$
- N_2 with columns indexed by $[n]$ and such that $\mathcal{L}_{N_2}=\varnothing$

Now, observe that

$$M_1 \cong \left(\begin{array}{ccc|c} & & & 1 \\ & M_2 & & \vdots \\ & & & 1 \\ \hline 1 & \dots & 1 & 0 \end{array}\right)$$

Moreover, by Lemma 6.3.2, $M_1 \in \mathrm{BR}(\mathcal{U}_{3,n})$ and $M_2 \in \mathrm{BR}(\mathcal{U}_{3,n-1})$. So $d(\mathcal{U}_{3,n}, \mathcal{U}_{3,n-1}) = 1$. On the other hand,

$$N_2 \cong \left(\begin{array}{ccc|c} & & & 1 \\ & N_1 & & \vdots \\ & & & 1 \\ \hline 1 & \dots & 1 & 0 \end{array}\right)$$

and clearly $N_1 \in \mathrm{BR}(\mathcal{U}_{2,n-1})$ and $N_2 \in \mathrm{BR}(\mathcal{U}_{2,n-1})$. So $d(\mathcal{U}_{2,n-1}, \mathcal{U}_{2,n}) = 1$. So

$$\begin{aligned} d(\mathcal{U}_{2,n}, \mathcal{U}_{3,n}) &\leq d(\mathcal{U}_{2,n}, \mathcal{U}_{2,n-1}) + d(\mathcal{U}_{2,n-1}, \mathcal{U}_{3,n-1}) + d(\mathcal{U}_{3,n-1} + \mathcal{U}_{3n}) \\ &= 1 + d(\mathcal{U}_{2,n-1}, \mathcal{U}_{3,n-1}) + 1 && \text{(by the previous computations)} \\ &\leq 1 + (2(n-1) - 5) + 1 && \text{(by the induction hypothesis)} \\ &= 2n - 5 \end{aligned}$$

□

6.4 Examples

Definition 6.4.1. Given a finite nonempty set V and a family $\mathcal{F} \subset P_3(V)$, define the simplicial complex $\mathcal{H}^V_{\mathcal{F}} = (V, H_{\mathcal{F}})$ given by $H_{\mathcal{F}} := P_{\leq 3}(V) \setminus \mathcal{F}$.

Lemma 6.4.2. *Given a finite nonempty set V and a PEG $\mathcal{F} \subset P_3(V)$, the simplicial complex $\mathcal{H}^V_{\mathcal{F}}$ is a matroid.*

Proof. Observe that $\mathcal{H}^V_{\mathcal{F}} \in \mathrm{Pav}_2$. By Proposition 4.2.5 it suffices to show that $P_3(V) \subseteq \mathrm{Fl}\,\mathcal{H}^V_{\mathcal{F}} \cup H_{\mathcal{F}}$. Consider then a set $X \in P_3(V) \setminus H_{\mathcal{F}} = \mathcal{F}$. Consider now a subset $I \in 2^X \cap H_{\mathcal{F}}$ and an element $p \in V \setminus X$. If $|I| \leq 1$ then $I \cup \{p\} \in P_{\leq 2}(V) \subseteq H_{\mathcal{F}}$. Recall now that the family $\mathcal{F}$ is a PEG. Therefore, if $|I| = 2$, then $I \cup \{p\} \notin \mathcal{F}$ and so $I \cup \{p\} \in H_{\mathcal{F}}$. Hence $X \in \mathrm{Fl}\,\mathcal{H}^V_{\mathcal{F}}$ and we may conclude that $P_3(V) \subseteq \mathrm{Fl}\,\mathcal{H}^V_{\mathcal{F}} \cup H_{\mathcal{F}}$. □

Definition 6.4.3. Given $k, n \in \mathbb{N}$ with $n \geq 2k + 1$ define sets $\lambda_i := \{1, 2i, 2i + 1\} \in P_3([n])$, for every $i \in [k]$, the family $\Lambda_k := \{\lambda_i \mid i \in [k]\}$ and the simplicial complex $\mathcal{S}_{k,n} := \mathcal{H}^{[n]}_{\Lambda_k} = ([n], P_{\leq 3}([n]) \setminus \Lambda_k)$.

Remark 6.4.4. Given $k, n \in \mathbb{N}$ with $n \geq 2k + 1$, $\mathcal{S}_{k,n}$ is a matroid.

Proof. Observe that the family Λ_k is a PEG and therefore the result follows from Lemma 6.4.2. □

Proposition 6.4.5. *Given $k, n \in \mathbb{N}$ with $n \geq 2k+1$, write $p := n-(2k+1)$ and consider a graph $\Gamma \in \mathcal{A}_{\mathcal{S}_{k,n}}$. Then*

$$e(\Gamma) \leq \left\lfloor \frac{p^2}{4} \right\rfloor + kp + 3k$$

Proof. Write $\mathcal{A}_p := \mathcal{A}_{\mathcal{S}_{k,n}}$ and $I(p) := \lfloor p^2/4 \rfloor + kp + 3k$. Proceed by induction on p to prove that $e(\Gamma) \leq I(p)$ for every graph $\Gamma \in \mathcal{A}_p$. Consider first a graph $\Gamma = (V, E) \in \mathcal{A}_0$ (every vertex lies in a triangle of the family Λ_k in such a graph). We claim that $E = \bigcup_{i \in [k]} P_2(\lambda_i)$. To prove the claim, observe that the inclusion $\bigcup_{i\in[k]} P_2(\lambda_i) \subseteq E$ holds clearly. Conversely, suppose that there exists an edge $X \in E \setminus \bigcup_{i\in[k]} P_2(\lambda_i)$. Then there exist distinct indices $i, j \in [k]$ such that one of the following conditions holds:

- $X = \{2i, 2j\}$
- $X = \{2i+1, 2j+1\}$
- $X = \{2i, 2j+1\}$

Then the subgraph $\Gamma[X \cup \{1\}]$ is a triangle of Γ, but $X \cup \{1\} \notin \Lambda_k$. This is a contradiction, because $\Gamma \in \mathcal{A}_0$. Hence the claim holds and therefore $e(\Gamma) = 3k = I(0)$.

Consider now $p \geq 1$, suppose that the number of edges of every graph in $\mathcal{A}_{p-1}$ is at most $I(p-1)$ and consider a graph $\Gamma = (V, E) \in \mathcal{A}_p$. Define the set $V' := V \setminus [2k+1]$ (the set of vertices not lying on a triangle of Γ) and the subgraph $\Gamma' := \Gamma[V']$. We claim that there exists a vertex $v_0 \in V'$ such that $\deg_{\Gamma'}(v_0) \leq \lfloor p/2 \rfloor$. To prove the claim, suppose that $\deg_{\Gamma'}(v) > p/2$ for every $v \in V'$. Then

$$e(\Gamma') = \frac{\sum_{v\in V'} \deg_{\Gamma'}(v)}{2} > \frac{|V'|\,p}{4} = \frac{p^2}{4} \geq \left\lfloor \frac{p^2}{4} \right\rfloor$$

This contradicts Theorem 6.2.6, because Γ' is a triangle-free graph with p vertices. Observe that the vertex v_0 does not lie on any triangle of Γ. Therefore v_0 is adjacent to at most one vertex of each triangle of Γ. So $|\mathcal{N}_\Gamma(v_0) \cap (V \setminus V')| \leq k$. Hence

$$\deg_\Gamma(v_0) = \deg_{\Gamma'}(v_0) + |\mathcal{N}_\Gamma(v_0) \cap (V \setminus V')| \leq \left\lfloor \frac{p}{2} \right\rfloor + k$$

Now, define the subgraph $\Gamma_0 := \Gamma[V \setminus \{v_0\}]$ and observe that $\Gamma_0 \in \mathcal{A}_{p-1}$. Hence:

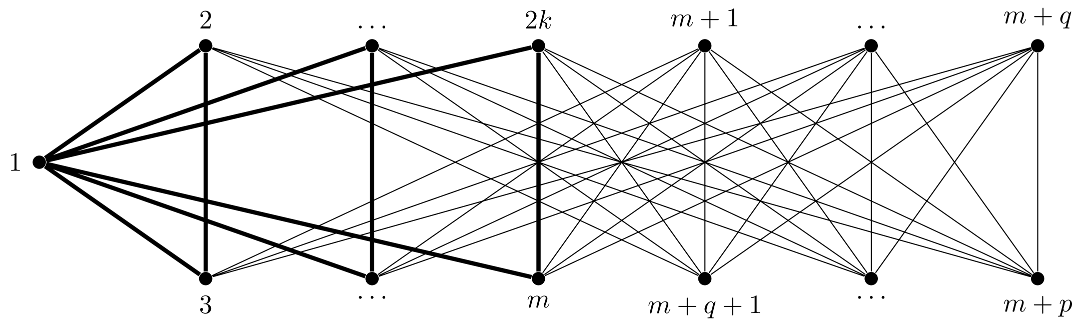
$$\begin{aligned}
e(\Gamma) &= e(\Gamma_0) + \deg_\Gamma(v_0) \\
&\leq \left(\left\lfloor \frac{(p-1)^2}{4} \right\rfloor + k(p-1) + 3k \right) + \deg_\Gamma(v_0) && \text{(by the induction hypothesis)} \\
&\leq \left(\left\lfloor \frac{(p-1)^2}{4} \right\rfloor + k(p-1) + 3k \right) + \left(\left\lfloor \frac{p}{2} \right\rfloor + k \right) && \text{(by the previous computation)} \\
&= \left(\left\lfloor \frac{(p-1)^2}{4} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor \right) + kp + 3k \\
&= \left\lfloor \frac{p^2}{4} \right\rfloor + kp + 3k && \text{(by Remark 6.1.9)}
\end{aligned}$$

□

Lemma 6.4.6. *Given $k, n \in \mathbb{N}$ with $n \geq 2k+1$, write $m := 2k+1$, $p := n-(2k+1)$ and $q := \lfloor p/2 \rfloor$. Define sets*

- $V_1 := \{2i \mid i \in [k]\}$
- $V_2 := \{2i+1 \mid i \in [k]\}$
- $V_3 := \{m+i \mid i \in [q]\}$
- $V_4 := \{m+i \mid i \in [p] \setminus [q]\}$
- $E := \left(\bigcup_{i\in[k]} P_2(\lambda_i)\right) \cup V_1V_4 \cup V_2V_3 \cup V_3V_4$

and the graph $\Gamma := ([n], E)$.



Then $\Gamma \in \mathcal{A}_{S_{k,n}}$ *and* $e(\Gamma) = \left\lfloor \frac{p^2}{4} \right\rfloor + kp + 3k$.

Proof. It is clear that the set of triangles of Γ is Λ_k so $\Gamma \in \mathcal{A}_{S_{k,n}}$. Moreover, the following equalities hold:

$$\begin{aligned} e(\Gamma) &= \left| \bigcup_{i\in[k]} P_2(\lambda_i) \right| + |V_1|\,|V_4| + |V_2|\,|V_3| + |V_3|\,|V_4| \\ &= 3k + k(p-q) + kq + q(p-q) \\ &= 3k + kp + \left\lfloor \frac{p}{2} \right\rfloor \left(p - \left\lfloor \frac{p}{2} \right\rfloor\right) \\ &= 3k + kp + \left\lfloor \frac{p^2}{4} \right\rfloor && \text{(by Remark 6.1.9)} \end{aligned}$$

□

Corollary 6.4.7. *Given* $n, k \in \mathbb{N}$ *with* $n \geq 2k+1$, *write* $p := n - (2k+1)$. *Then*

$$d_n(\mathcal{U}_{2,n}, \mathcal{S}_{k,n}) \leq \binom{n}{2} - \left(\left\lfloor \frac{p^2}{4} \right\rfloor + kp + 3k \right)$$

Proof.

$$\begin{aligned} d_n(\mathcal{U}_{2,n}, \mathcal{S}_{k,n}) &\leq \binom{n}{2} - \max_{\Gamma\in\mathcal{A}_{\mathcal{S}_{k,n}}} e(\Gamma) && \text{(by Lemma 6.1.8)} \\ &= \binom{n}{2} - \left(\left\lfloor \frac{p^2}{4} \right\rfloor + kp + 3k \right) && \text{(by Proposition 6.4.5 and Lemma 6.4.6)} \end{aligned}$$

□

Definition 6.4.8. Given $k, n \in \mathbb{N}$ with $n \geq 3k$ define sets $\pi_i := \{3i-2, 3i-1, 3i\} \in P_3([n])$, for every $i \in [k]$, the family $\Pi_k := \{\pi_i \mid i \in [k]\}$ and the simplicial complex $\mathcal{T}_{k,n} := \mathcal{H}^{[n]}_{\Pi_k} = ([n], P_{\leq 3}([n]) \setminus \Pi_k)$.

Remark 6.4.9. Given $k, n \in \mathbb{N}$ with $n \geq 3k$, $\mathcal{T}_{k,n}$ is a matroid.

Proof. Observe that the family Π_k is a PEG and therefore the result follows from Lemma 6.4.2. ☐

Proposition 6.4.10. *Given $k, n \in \mathbb{N}$ with $n \geq 3k$, write $p := n - 3k$ and consider a graph $\Gamma \in \mathcal{A}_{\mathcal{T}_{k,n}}$. Then*

$$e(\Gamma) \leq \left\lfloor \frac{p^2}{4} \right\rfloor + kp + 3k + 3\binom{k}{2}$$

Proof. Write $\mathcal{A}_p := \mathcal{A}_{\mathcal{T}_{k,3k+p}}$ and $I(p) := \lfloor p^2/4 \rfloor + kp + 3k + 3\binom{k}{2}$. Proceed by induction on p to prove that $e(\Gamma) \leq I(p)$ for every graph $\Gamma \in \mathcal{A}_p$. Consider first a graph $\Gamma \in \mathcal{A}_0$ (every vertex lies in a triangle of the family Π_k in such a graph). Observe that, for all distinct indices $i, j \in [k]$, every vertex $v \in \pi_j$ is adjacent to at most 1 vertex of π_i; and therefore there are at most 3 edges joining π_i and π_j. Therefore we may conclude that

$$\begin{aligned} e(\Gamma) &\leq \left| \bigcup_{i \in [k]} P_2(\pi_i) \right| + 3\binom{k}{2} \\ &= 3k + 3\binom{k}{2} = I(0) \end{aligned}$$

Consider now $p \geq 1$, suppose that the number of edges of every graph in $\mathcal{A}_{p-1}$ is at most $I(p-1)$ and consider a graph $\Gamma = (V, E) \in \mathcal{A}_p$. Define the set $V' := V \setminus [3k]$ (the set of vertices not lying on a triangle of Γ) and the subgraph $\Gamma' := \Gamma[V']$. We claim that there exists a vertex $v_0 \in V'$ such that $\deg_{\Gamma'}(v_0) \leq \lfloor p/2 \rfloor$. To prove the claim, suppose that $\deg_{\Gamma'}(v) > p/2$ for every $v \in V'$. Then

$$e(\Gamma') = \frac{\sum_{v \in V'} \deg_{\Gamma'}(v)}{2} > \frac{|V'|\,p}{4} = \frac{p^2}{4} \geq \left\lfloor \frac{p^2}{4} \right\rfloor$$

This contradicts Theorem 6.2.6, because Γ' is a triangle-free graph with p vertices. Observe that the vertex v_0 does not lie on any triangle of Γ. Therefore v_0 is adjacent to at most one vertex of each triangle of Γ. So $|\mathcal{N}_\Gamma(v_0) \cap (V \setminus V')| \leq k$. Hence

$$\deg_\Gamma(v_0) = \deg_{\Gamma'}(v_0) + |\mathcal{N}_\Gamma(v_0) \cap (V \setminus V')| \leq \left\lfloor \frac{p}{2} \right\rfloor + k$$

Now, define the subgraph $\Gamma_0 := \Gamma[V \setminus \{v_0\}]$ and observe that $\Gamma_0 \in \mathcal{A}_{p-1}$. Hence:

$$\begin{aligned} e(\Gamma) &= e(\Gamma_0) + \deg_\Gamma(v_0) \\ &\leq \left(\left\lfloor \frac{(p-1)^2}{4} \right\rfloor + k(p-1) + 3k + 3\binom{k}{2} \right) + \deg_\Gamma(v_0) && \text{(by the induction hypothesis)} \\ &\leq \left(\left\lfloor \frac{(p-1)^2}{4} \right\rfloor + k(p-1) + 3k + 3\binom{k}{2} \right) + \left(\left\lfloor \frac{p}{2} \right\rfloor + k \right) && \text{(by the previous computation)} \\ &= \left(\left\lfloor \frac{(p-1)^2}{4} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor \right) + kp + 3k + 3\binom{k}{2} \\ &= \left\lfloor \frac{p^2}{4} \right\rfloor + kp + 3k + 3\binom{k}{2} && \text{(by Remark 6.1.9)} \end{aligned}$$

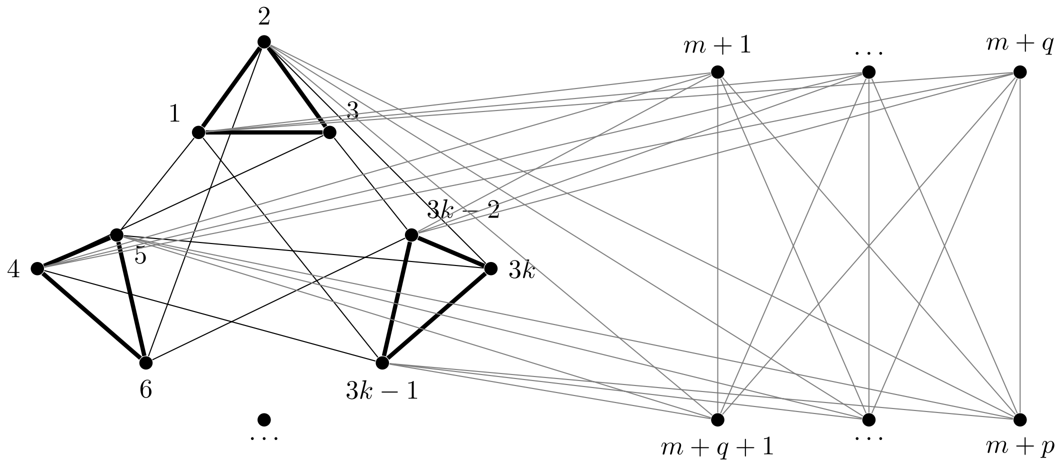
☐

Lemma 6.4.11. *Given $k, n \in \mathbb{N}$ with $n \geq 3k$, write $m := 3k$, $p := n - 3k$ and $q := \lfloor p/2 \rfloor$. Define sets*

- $V_1 := \{3i - 2 \mid i \in [k]\}$

- $V_2 := \{3i-1 \mid i \in [k]\}$
- $V_3 := \{m+i \mid i \in [q]\}$
- $V_4 := \{m+i \mid i \in [p] \setminus [q]\}$
- $E_0 := \{\{3i-2, 3j-1\}, \{3i-1, 3j\}, \{3i, 3j-2\} \mid i, j \in [k]$ *with* $i < j\}$
- $E := \left(\bigcup_{i\in[k]} P_2(\pi_i)\right) \cup E_0 \cup V_1V_4 \cup V_2V_3 \cup V_3V_4$

and the graph $\Gamma := ([n], E)$*.*



Then $\Gamma \in \mathcal{A}_{T_{k,n}}$ *and* $e(\Gamma) = \left\lfloor \frac{p^2}{4} \right\rfloor + kp + 3k + 3\binom{k}{2}$*.*

Proof. For every index $i \in [k]$, the following equalities hold:

- $\mathcal{N}_\Gamma(3i-2) = \{3i-1, 3i, 3j-1, 3\ell \mid \ell < i < j\}$
- $\mathcal{N}_\Gamma(3i-1) = \{3i-2, 3i, 3j, 3\ell-2 \mid \ell < i < j\}$
- $\mathcal{N}_\Gamma(3i) = \{3i-2, 3i-1, 3j-2, 3\ell-1 \mid \ell < i < j\}$

So, for all indices $\ell, i, j \in [k]$ with $\ell < i < j$, we may conclude that:

1. $\{3i-1, 3j-1\}, \{3i-1, 3\ell\}, \{3i, 3j\}, \{3i, 3\ell\}, \{3j-1, 3\ell\} \notin E$
2. $\{3i-2, 3j\}, \{3i-2, 3\ell-2\}, \{3i, 3j\}, \{3i, 3\ell-2\}, \{3j, 3\ell-2\} \notin E$
3. $\{3i-2, 3j-2\}, \{3i-2, 3\ell-1\}, \{3i-1, 3j-2\}, \{3i-1, 3\ell-1\}, \{3j-2, 3\ell-1\} \notin E$

Hence, by condition 1, the only triangle containing the vertex $3i-2$ is π_i; by condition 2, the only triangle containing the vertex $3i-1$ is π_i; and, by condition 3, the only triangle containing the vertex $3i$ is π_i. Hence the set of triangles of Γ is Π_k so $\Gamma \in \mathcal{A}_{T_{k,n}}$. Moreover the set E_0 contains exactly 3 edges joining each pair of triangles, so the following equalities hold:

$$
\begin{aligned}
e(\Gamma) &= \left| \bigcup_{i\in[k]} P_2(\lambda_i) \right| + |E_0| + |V_1|\,|V_4| + |V_2|\,|V_3| + |V_3|\,|V_4| \\
&= 3k + 3\binom{k}{2} + k(p-q) + kq + q(p-q) \\
&= 3k + 3\binom{k}{2} + kp + \left\lfloor \frac{p}{2} \right\rfloor \left(p - \left\lfloor \frac{p}{2} \right\rfloor\right) \\
&= 3k + 3\binom{k}{2} + kp + \left\lfloor \frac{p^2}{4} \right\rfloor && \text{(by Remark 6.1.9)}
\end{aligned}
$$

□

Corollary 6.4.12. *Given $n, k \in \mathbb{N}$ with $n \geq 3k$, write $p := n - 3k$. Then*

$$d_n(\mathcal{U}_{2,n}, \mathcal{T}_{k,n}) \leq \binom{n}{2} - \left(\left\lfloor \frac{p^2}{4} \right\rfloor + kp + 3k + 3\binom{k}{2} \right)$$

Proof.

$$\begin{aligned} d_n(\mathcal{U}_{2,n}, \mathcal{T}_{k,n}) &\leq \binom{n}{2} - \max_{\Gamma \in \mathcal{A}_{\mathcal{T}_{k,n}}} e(\Gamma) && \text{(by Lemma 6.1.8)} \\ &= \binom{n}{2} - \left(\left\lfloor \frac{p^2}{4} \right\rfloor + kp + 3k + 3\binom{k}{2} \right) && \text{(by Proposition 6.4.10 and Lemma 6.4.11)} \end{aligned}$$

□

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